JCARME
ISSN: 2228-7922
Journal of Computational and Applied Research in Mechanical Engineering
Vol. 5, No. 1, pp. 25- 35, Autumn-Winter 2015-16
jcarme.srttu.edu

Assessing different nonlinear analysis methods for free vibrations of initially stressed composite laminated plates

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Article info:
Received: 12/01/2015
Accepted: 03/08/2015
Online: 11/09/2015

Keywords:
Nonlinear vibrations,
Biaxial initial stress,
Homotopy perturbation method,
Multiple scales perturbation method.

Abstract
In this paper, the nonlinear free vibrations of thin symmetric and non-symmetric cross-ply composite plates subjected to biaxial initial stresses are investigated. Because of their excellent properties such as specific strength and specific stiffness, composite plates have wide applications in aerospace and mechanical structures. Based on Von-Karman's strain-displacement relations and using Galerkin method, the nonlinear differential equation of free vibrations of initially stressed composite plate is obtained. This nonlinear equation is solved using two different analytical perturbation methods, namely method of multiple scales (MTS) and homotopy perturbation method (HPM), to analyze the nonlinear vibrations of initially stressed cross-ply composite plates. Effects of tensile and compressive biaxial initial stresses, initial vibration amplitude, thickness, and aspect ratios of the composite plates on the frequency behavior are investigated. The validity of the results is confirmed by making a comparison with those reported in the literature. According to the results, both analytical solutions show increasing trends for natural frequency parameters by increasing normal initial stresses. Regardless of the value of initial biaxial stresses, for both symmetric and non-symmetric plates, the results of MTS and HPM are in close agreement for the smallest initial amplitude. However, for compressive initial stresses, by increasing initial amplitude ratios, the discrepancies between the results of HPM and MTS increase for symmetric and non-symmetric plates. Although HPM includes less computational effort (smaller length of formulation) than MTS, the linear-to-nonlinear frequency ratios obtained using MTS method become closer to those obtained by HPM as initial vibration amplitude is decreased and initial stress is increased.

1. Introduction

Because of their high strength and stiffness to weight ratio, composite laminated plates are widely used in many high technology industries such as aerospace, aircraft, and automobile structures. Dynamic behaviors of these composite plates arealways very vital for researchers in order to find an appropriate material with the best response to the dynamic conditions. Since the accuracy of results in

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dynamic analysis is so important, researchers use different methods to find linear and nonlinear results of the vibration of composite plates. Among different kinds of composite plates, cross-ply composite plates have an important position owing to their special strength and stiffness properties. Different kinds of materials are used to construct composite plates. The outstanding properties of composite materials such as thermal insulation, wear resistance, and high strength and stiffness to weight ratios have attracted considerable attention in recent years. Singh et al. [1] used the method of direct numerical integration of the frequency ratio in the study of nonlinear free vibration behavior of rectangular cross-ply laminates. Radu [2] discussed Singh's analysis and modified its formulation according to the new equations. Zhen et al. [3] used global-local higher-order theory for the free vibration of laminated composite and sandwich plates. A nonlinear elastic foundation was applied by Chien et al. [4] to analyze the nonlinear vibration of laminated plates. Matsunaga [5] investigated the free vibrations of cross-ply composite shells subjected to in-plane stresses using higher-order shear deformation theory. Nonlinear free vibrations of laminated composite plates on the elastic foundation and random system properties were investigated by Lal et al. [6]. A closed form solution was done for linear and nonlinear free vibrations of composite and fiber metal laminated plates using Galerkin and multiple scales solution methods by Shooshtari et al. [7].

The existence of initial stresses during the application of composite plates is very common in real applications, especially in aerospace structures. Consequently, researchers have a special concern about this important issue and perform some analyses with initial stresses applying to different composite structures.

Chen et al. [8] considered an initial stress in the analysis of nonlinear vibrations of hybrid composite plates. The dynamic response of initially stressed functionally graded material thin plate was investigated by Yang et al. [9]. Chen et al. [10] conducted the nonlinear analysis of initially stressed composite plates using Runge-Kutta method to solve the main nonlinear equation.

Different methods exist to find an appropriate response of nonlinear dynamic equations such as numerical method of Lindstedt-Poincare, averaging method, homotopy perturbation method (HPM), and multiple scales perturbation technique (MTS).

Recently, HPM as an easy analytical tool for solving nonlinear problems has received considerable attention. A new perturbation technique coupled with homotopy method to solve nonlinear equations was presented by He [11]. An eigenvalue problem of a lined duct using homotopy method was studied by Sun [12]. Blendez [13] used He's HPM to show an approximate solution for conservative nonlinear oscillations. A nonlinear vibration analysis was investigated by Yazdi [14] on the FGM plate using homotopy perturbation technique. Yongqiang [15] analyzed nonlinear free vibrations of symmetric rectangular honeycomb sandwich panels using HPM.

Nowadays, MTS is a well-developed, well-accepted, and very popular method to approximate solutions of nonlinear differential equations. This method was developed from 1935 to 1962 by Krylov and Bogoliubov, Kuzmak, Kevorkian, and Cole, Cochran, and Mahony. In early 1970s, Nayfeh developed this method by writing many papers and books on this subject. Usefulness and importance of the MTS is related to the fact that the governing mechanics of deforming media occur over the course of time. Also, MTS has good computational power to solve nonlinear differential equations. Bhimaraddi [16] used multiple scales to solve the nonlinear flexural vibrations of rectangular plates subjected to in-plane forces using a new shear deformation theory. Buckling and post-buckling of extensible rods were investigated by Mazzilli [17] using a multiple scales solution.

In this paper, the geometrically nonlinear vibration of thin composite cross-ply plates subjected to the biaxial tensile and compressive initial stresses is studied. HPM and MTS perturbation techniques are used to compare two analytical solutions in terms of accuracy and computational efficiency. Biaxial initial stresses, maximum vibration amplitude, thickness of the composite plates, and aspect ratio of composite plates are considered the parameters that affect the frequency ratio. New results are presented as benchmark solutions in the state of the art.

2. Problem description

A multilayered thin composite plate with length *a*, width *b*, and uniform thickness *h* is considered in Fig. 1. *u*, *v*, and *w* are the displacement components of the plate in *x*, *y*, and *z* directions, respectively. The origin of the Cartesian coordinate system is placed on the mid-surface of the plate.

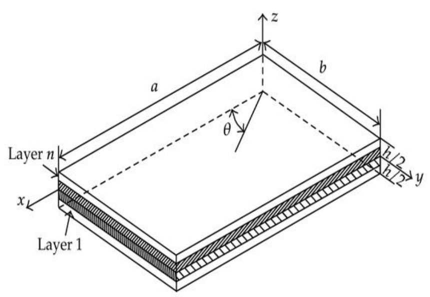

Fig. 1. Geometry of thin composite plate.

According to the assumptions of large-amplitude deformation of the plate in plane-stress state, the dynamic Von-Karman strain-displacement relations are employed [1]:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \tag{1}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \tag{2}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) - 2z\frac{\partial^2 w}{\partial x \partial y} \tag{3}$$

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0 \tag{4}$$

Based on the classical lamination theory (CLT) and considering plane-stress conditions, the stress-strain relations are defined as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{pmatrix} \overline{Q_{11}} & \overline{Q_{12}} & 0 \\ \overline{Q_{12}} & \overline{Q_{22}} & 0 \\ 0 & 0 & \overline{Q_{66}} \end{pmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{5}$$

where $\overline{Q_{ij}}$ are elements of reduced stiffness matrix for each ply, defined in Appendix A. The forces (N_x, N_y, N_{xy}), moment resultants (M_x, M_y, M_{xy}),pre-loads including in-plane forces (N_x^i, N_y^i, N_{xy}^i), and out-of-plane moments (M_x^i, M_y^i, M_{xy}^i) applied to the plate are given by:

$$\begin{Bmatrix} N_x + N_x^i \\ N_y + N_y^i \\ N_{xy} + N_{xy}^i \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} + \sigma_{xx}^i \\ \sigma_{yy} + \sigma_{yy}^i \\ \sigma_{xy} + \sigma_{xy}^i \end{Bmatrix} dz \tag{6}$$

$$\begin{Bmatrix} M_x + M_x^i \\ M_y + M_y^i \\ M_{xy} + M_{xy}^i \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} + \sigma_{xx}^i \\ \sigma_{yy} + \sigma_{yy}^i \\ \sigma_{xy} + \sigma_{xy}^i \end{Bmatrix} z dz \tag{7}$$

If the initial stresses are considered uniform through the thickness, only in-plane initial stresses (force pre-loads) affect the nonlinear vibration of composite plate and the moment pre-loads become zero. Hence, considering in-plane initial stresses only, the governing equations of the system are obtained using the Hamilton's principle [1]:

$$\frac{\partial (N_x + N_x^i)}{\partial x} + \frac{\partial (N_{xy} + N_{xy}^i)}{\partial y} = 0 \tag{8}$$

$$\frac{\partial (N_y + N_x^i)}{\partial y} + \frac{\partial (N_{xy} + N_{xy}^i)}{\partial x} = 0 \tag{9}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}\left((N_x + N_x^i)\frac{\partial w}{\partial x}\right)$$
$$+ \frac{\partial}{\partial x}\left((N_{xy} + N_{xy}^i)\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left((N_{xy} + N_{xy}^i)\frac{\partial w}{\partial x}\right)$$
$$+ \frac{\partial}{\partial y}\left((N_y + N_y^i)\frac{\partial w}{\partial y}\right) = I_0 \frac{\partial^2 w}{\partial t^2} \tag{10}$$

where

$$I_0 = \int_{-h/2}^{h/2} \rho(z) dz \tag{11}$$

The boundary conditions are considered simply supported defined as follows [1]:

$$x = 0, a\ ,\ v = w = 0\ , N_x = M_x = 0 \quad (12)$$

$$y = 0, b\ ,\ u = w = 0\ , N_y = M_y = 0 \quad (13)$$

Here, the set of admissible functions which satisfy the simply supported boundary conditions are considered for the displacement components to solve the nonlinear free vibration problem[1]:

$$u = U(t)\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \quad (14)$$

$$v = V(t)\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \quad (15)$$

$$w = W(t)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y \quad (16)$$

U(*t*), *V*(*t*), and *W*(*t*) are the displacement amplitudes in terms of generalized coordinate, *t*. By substituting the displacement components, Eqs. (14)-(16), in the governing equations (Eqs. (8)-(10)), applying Galerkin method, and simplification, *W*(*t*) is found in terms of *U*(*t*) and *V*(*t*) according to a single second-order ordinary differential nonlinear equation as follows:

$$\frac{d^2W(t)}{dt^2} + \alpha W(t) + \beta W(t)^2 + \gamma W(t)^3 = 0 \quad (17)$$

where α, β and γ are constant coefficients depending on the stiffness and mass of the composite plate, mentioned in Appendix B. The initial conditions for the lateral deflection of the plate are considered as follows:

$$W(0) = W_{\max}\ ,\ \frac{dW(0)}{dt} = 0 \quad (18)$$

where $W_{\max}$ stands for the maximum vibration amplitude of the composite plate.

3. Analytical solutions

In order to solve Eq. (17) as a nonlinear differential equation, many methods can be applied to get the approximate analytical result. In this paper, HPM and MTS perturbation methods are considered and discussed by numerical results to find the efficiency of each analytical method.

3. 1. HPM solution procedure

HPM is used here to find the frequency results of the considered cross-ply composite plate. $w(r,p): \Omega \times [0:1]$ defined as a homotopy small parameter which satisfies the following expression is constructed [11]:

$$H(w,p) = L(w) - L(u_0) + pL(u_0) + pN(w) = 0 \quad (19)$$

Linear and nonlinear parts of Eq. (17) are separated by *L* and *N*, respectively, and u_0 is the first approximation which satisfies the initial conditions [11]. Solution of Eq. (17) is assumed as a power series [11]:

$$w = w_0 + pw_1 + p^2w_2 + ... \quad (20)$$

Substitution of Eq. (20) in Eq. (19) yields:

$$L(w_0) - L(u_0) = 0 \quad (21)$$

$$L(w_1) + L(u_0) + \beta w_0^2 + \gamma w_0^3 = 0 \quad (22)$$

for which the initial conditions are obtained from Eq. (18):

$$w_0 = W_{\max}\ ,\ \frac{dw_0}{dt} = 0 \quad (23)$$

and,

$$w_1(0) = \frac{dw_1(0)}{dt} = 0 \quad (24)$$

The first approximation u_0 is guessed as follows to satisfy the initial conditions[11]:

$$w_0(t) = u_0(t) = \mathrm{W}_{\max}\cos(\omega t) \quad (25)$$

So from Eq. (20) we have:

$$\frac{d^2w_1}{dt^2} + \alpha w_1 + W_{\max}(\alpha - \omega^2 + \frac{3}{4}W_{\max}^2\gamma)\cos(\omega t) + \frac{\beta W_{\max}^2}{2}\cos(2\omega t) + \frac{\gamma W_{\max}^3}{4}\cos(3\omega t) + \frac{\beta W_{\max}^2}{2} = 0 \quad (26)$$

Using variational iteration method [14], the solution of Eq. (26) can be obtained. Hence:

$$w_1(t) = \frac{1}{\alpha}\int_0^t \sin(\tau - t) [W_{\max}(\alpha - \omega^2 + \frac{3}{4}W_{\max}^2\gamma)\cos(\omega t) + \frac{\beta W_{\max}^2}{2}\cos(2\omega t) + \frac{\gamma W_{\max}^3}{4}\cos(3\omega t) + \frac{\beta W_{\max}^2}{2}] \quad (27)$$

By integrating Eq. (27), the solution is:

$$w_1(t) = W_{max}(\alpha-\omega^2+\frac{3}{4}W_{max}^2\gamma)(\frac{1}{\alpha(\omega^2-1)})\cos\omega t +W_{max}^2(\frac{\beta}{2})(\frac{1}{\alpha(4\omega^2-1)})\cos(2\omega t) + W_{max}^3(\frac{\gamma}{4})(\frac{1}{\alpha(9\omega^2-1)})\cos(3\omega t) - \frac{\beta W_{max}^2}{2\alpha} +[W_{max}\frac{(-\alpha+\omega^2-\frac{3}{4}W_{max}^2\gamma)}{\alpha(\omega^2-1)} - \frac{\beta W_{max}^2}{2\alpha(4\omega^2-1)} - W_{max}^3(\frac{\gamma}{4})(\frac{1}{\alpha(9\omega^2-1)}) + \frac{\beta W_{max}^2}{2\alpha}]\cos(t) \quad (28)$$

In order to find ω, the coefficient of the last term, i.e. secular term, in Eq. (28) should be zero. Therefore, a sixth-order ordinary differential equation in terms of ω is obtained whose coefficients contain the geometric and stiffness parameters of composite cross-ply plate.

Finally, the first-order approximation of Eq. (17) is obtained as follows:

$$W(t) \approx w_0(t) + w_1(t) = W_{max}\left(\frac{\omega^2(\alpha-1)+{}^3\!/_4 W_{max}^2\gamma}{\alpha(\omega^2-1)}\right)\cos(\omega t) +W_{max}^2(\frac{\beta}{2})(\frac{1}{\alpha(4\omega^2-1)})\cos(2\omega t) + W_{max}^3(\frac{\gamma}{4})(\frac{1}{\alpha(9\omega^2-1)})\cos(3\omega t) - \frac{\beta W_{max}^2}{2\alpha} \quad (29)$$

3. 2. MTS solution procedure

Multiple scales perturbation method is used here to find the frequency results of the considered composite plate. Solution of Eq. (17) is written as an asymptotic expansion and spatial scales are introduced instead of time scales:

$$w(t) = w(T_0,T_1,T_2,...) = \varepsilon w_1(T_0,T_1,T_2,...) +\varepsilon^2 w_2(T_0,T_1,T_2,...) +\varepsilon^3 w_3(T_0,T_1,T_2,...) + ... \quad (30)$$

$$T_n = \varepsilon^n t\,,\, n = 0,1,2,... \quad (31)$$

The function $w_q = w_q(T_0,T_1,T_2,...)\,,\, q = 1,2,...$ is calculated when coefficients of increasing orders of ε, extracted from Eq. (17), are solved and secular terms are eliminated. Derivative operators are given below:

$$\frac{d}{dt} = D_0 + D_1\varepsilon + D_2\varepsilon^2 \quad (32)$$

$$\frac{d^2}{dt^2} = D_0{}^2 + 2D_0D_1\varepsilon + (D_1{}^2 + 2D_0D_2)\varepsilon^2 + ... \quad (33)$$

By substituting Eqs. (30)-(33) in Eq. (17) and retaining the terms of the coefficients of the first-order of ε, the following equation is obtained:

$$D_0{}^2 w_1 + \alpha w_1 = 0 \quad (34)$$

Solution of Eq. (34) can be written in a complex form as follows:

$$w_1 = Ae^{i\sqrt{\alpha}T_0} + \text{ComplexConjugate} \quad (35)$$

where A is a complex function of T_1, T_2, T_3.

By retaining the terms of the coefficients of the second-order of ε, the following equation is obtained:

$$D_0{}^2 w_2 + \alpha w_2 = -2D_0D_1w_1 - \beta w_1{}^2 \quad (36)$$

For the convergence of expansion Eq. (30), the secular term on the right-hand side of Eq. (36) must be eliminated, which leads to the so-called solubility condition equation:

$$D_1 w_1 = 0 \Rightarrow A = A(T_2, T_3, ...) \quad (37)$$

and

$$w_2 = \beta\frac{A^2}{3\omega_L^2}e^{2i\sqrt{\alpha}T_0} - \beta\frac{A\bar{A}}{\omega_L^2} + \text{C.C.} \quad (38)$$

Retaining the terms of the coefficients of the third-order of ε, the following equation is obtained :

$$D_0{}^2 w_3 + \alpha w_3 = -2D_0D_1w_2 - 2D_0D_2w_1 -D_1^2 w_1 - 2w_1w_2\beta - \gamma w_1^3 \quad (39)$$

In order to eliminate the secular term, the following solubility condition equation is obtained:

$$-2i\sqrt{\alpha}D_2A + \frac{10\beta^2}{3\alpha}A^2\bar{A} - 3\gamma A^2\bar{A} = 0 \quad (40)$$

where $\overline{A}$ is the complex conjugate (C.C.) of A. Solution of Eq. (40) is considered in the following form:

$$A = \frac{1}{2} a e^{iB}, a, B \in R \quad (41)$$

where a is a constant coefficient and B is obtained as follows:

$$B = \left(\frac{9\gamma\alpha - 10\beta^2}{24\alpha^{(3/2)}} \right) a^2 T_2 + B_0 \quad (42)$$

A particular solution of Eq. (39) is:

$$w_3 = (\frac{2\beta^2 + 3\gamma\alpha}{24\alpha^2}) A^3 e^{3i\sqrt{\alpha}T_0} + \text{C.C.} \quad (43)$$

Hence, the natural frequency is obtained as follows:

$$\omega = \sqrt{\alpha}(1 + \frac{9\gamma\alpha - 10\beta^2}{24\alpha^2}(\varepsilon a_0)^2) \quad (44)$$

Finally, from Eqs. (35), (38) and (43), the deflection response up to terms of order ε^3 is obtained as follows:

$$w(t) = \varepsilon(Ae^{i\sqrt{\alpha}T_0} + \text{C.C.}) + \varepsilon^2(\frac{\beta A^2 e^{2i\sqrt{\alpha}T_0}}{3\alpha} - \frac{\beta}{\sqrt{\alpha}} A\overline{A} + \text{C.C}) + \varepsilon^3(A^3(\frac{2\beta^2 + 3\gamma\alpha}{24\alpha^2})e^{3i\sqrt{\alpha}T_0} + \text{C.C.}) \quad (45)$$

In the next section, both HPM and MTS are discussed in detail by some numerical solutions for two multilayered cross-ply composite plates.

4. Numerical results

The numerical results for cross-ply composite plates subjected to different initial stresses, initial amplitude, aspect ratios of the plate, and length to thickness ratio are provided. In order to validate both of the proposed HPM and MTS, the nonlinear vibration of thin cross-ply laminated plates with different ratios of initial stresses is compared with the one reported by Singh [1], Radu [2], and Bhimaraddi [16], as indicated in Tables 1 and 2, respectively. The values of initial biaxial stresses are considered:

$$N_x^i = N_y^i = N, N_{xy}^i = 0 \quad (46)$$

In the case of compressive initial stresses, the critical value of N is obtained and called, N_c^b, the critical buckling stress. The non-dimensional ratio of the biaxial initial stresses to the critical buckling stress (N/N_c^b) is considered in Tables 1 to 4.

The accuracy of HPM for nonlinear vibrations of cross-ply plates without initial stresses is investigated in Table 1. Results for HPM are compared with those by both Singh [1] and Radu [2] for two different lay-ups of cross-ply composite plates, aspect ratios, and initial amplitudes. Numerical integration method is used to find the nonlinear vibrations of the considered plates by Singh [1]. Radu [2] presented a discussion on Singh's results and modified his equations by the same numerical integration method.

According to Table 1, for both symmetric and non-symmetric cross-ply laminated plates, there is a good agreement between HPM results and those reported in the literature for small initial amplitudes. For instance, the greatest percentage discrepancy for W_{max}/h =0.25 is *0.8%*. However, by increasing the initial amplitude, the percentage discrepancies are increased and HPM shows less accuracy at high initial amplitudes like W_{max}/h =2 by the percentage discrepancy of 2.95% for the non-symmetric cross-ply plate.

Linear to nonlinear frequency ratios for two-layered cross-ply square plate that are calculated by both HPM and MTS for different initial stresses and initial amplitudes are compared with those reported by Bhimaraddi [16] in Table 2. As can be seen in this table, the maximum percentage discrepancy between the MTS results and those of Bhimaraddi [16] for initial tensile stresses is 3.06%, which occurs in the case of stress ratio (N/N_c^b) equal to 0.5 and initial amplitude (W_{max}/h) of 1. However, for compressive initial stresses, MTS shows less accuracy. The maximum percentage discrepancy is equal to -18.5%, which occurs for stress ratio (N/N_c^b) of -0.125 and initial amplitude (W_{max}/h) of 1.

According to Table 2, as compared to Bhimaraddi's [16] results for both HPM and MTS, the discrepancies are increased by increasing (W_{max}/h) and decreasing (N/N_c^b). Also, the discrepancies are decreased by decreasing (W_{max}/h) and increasing (N/N_c^b). Generally, the discrepancies corresponding to MTS are less than those for HPM, as compared to Bhimaraddi's [16] results.
The results of frequency ratio for both symmetric and non-symmetric cross-ply composite plates under different initial biaxial stresses and initial amplitudes are obtained in Table 3. According to this table, regardless of the value of initial biaxial stresses, for both symmetric and non-symmetric plates, the results of MTS and HPM are in close agreement for the smallest initial amplitude, W_{max}/h=0.25. However, for compressive initial stresses, $N/N_c^b = -0.5$, by increasing initial amplitude ratio from 0.25 to 1, the discrepancies between the results of HPM and MTS increase from 0.67% to 33.74% for the symmetric plate and 1.81% to 36.03% for non-the symmetric plate (*a/h*=75). Nevertheless, by altering biaxial initial stresses from compressive to the tensile one, lower discrepancies are obtained due to increasing initial amplitudes. For $N/N_c^b = 2$, by increasing initial amplitudes from 0.25 to 1, the discrepancies between HPM and MTS increase from 0.01% to 2.98% for the symmetric plates and 0.02% to 3.47% for the non-symmetric plates. According to this table, more accurate results are obtained for both composite plates due to tensile initial stresses. In contrast, for compressive initial stresses, less accurate results are shown for HPM and more reliable results from MTS are provided as compared to Bhimaraddi's [16] results. Also, it should be mentioned that symmetric plates show less discrepancies between MTS and HPM in comparison with the non-symmetric plates.

Table 1. Comparing nonlinear to linear frequency ratio (ω_{NL}/ω_L) for thin cross-ply laminated plate, $(0/90)_2$ and $(0/90)_s$, *a/b*=1, no initial stress (N/N_c^b) = 0. $E_1/E_2 = 10, G_{12}/E_2 = 0.5, \nu_{12} = 0.25,\ a/h = 75$, *n*=*m*=1.

	$(0/90)_s$			$(0/90)_2$		
W_{max}/h	**Singh[1]**	**Radu[2]**	**HPM**	**Singh[1]**	**Radu[2]**	**HPM**
0.25	1.0498	1.0629	1.0591(0.80)* (-0.3)⁺	1.0556	1.0702	1.0634(0.7) (-0.6)
0.5	1.1907	1.2306	1.2194(2.41) (-0.9)	1.2113	1.2552	1.2344(1.90) (-1.65)
1	1.6314	1.7400	1.7173(5.26) (-1.3)	1.6906	1.8080	1.7598(4.09) (-2.66)
2	2.7553	3.0052	2.9676(7.70) (-1.2)	2.8941	3.1609	3.0675(5.99) (-2.95)
***n*=1, *m*=2, (ω_{NL}/ω_L)**						
0.25	1.0441	1.0470	1.0441 (0) (-0.2)	1.0729	1.0704	1.0699(-0.2) (-0.04)
0.5	1.1701	1.1754	1.666(-0.2) (-0.7)	1.2718	1.2681	1.2566(-1.1) (-0.9)
1	1.5711	1.5826	1.5635(-0.4) (-1.2)	1.8589	1.8599	1.8214(-2.0) (-2.07)
2	2.6118	2.6363	2.6051(-0.2) (-1.1)	3.2804	3.2978	3.2066(-2.2) (-2.76)

*: Percentage discrepancy= (HPM-Ref [1])/Ref [1] *100
+: Percentage discrepancy= (HPM-Ref [2])/Ref [2]*100

Table 2. Comparing linear to nonlinear frequency ratio (ω_L / ω_{NL}) for thin two-layered cross-ply laminated plate, (0/90), *a/b*=1, $E_1/E_2 = 25, G_{12}/E_2 = 0.5, \nu_{12} = 0.25, \quad a/h = 10$, *n*=*m*=1.

N/N_c^b		**0.25**	**0.5**	**0.75**	**1**
	HPM	0.7642 (-17.3)*	0.5092 (-32.8)	0.3665 (-38.4)	0.2830 (-33.0)
-0.5	MTS	0.9272 (0.2)+	0.7611 (0.3)	0.5860 (-1.5)	0.4433 (4.8)
	Bim[16]	0.9246	0.7582	0.5953	0.4227
	HPM	0.8431(-11.6)	0.6164 (-27.2)	0.4622 (-37.7)	0.3637 (-49.1)
-0.125	MTS	0.9571(0.2)	0.8479 (0.1)	0.7125 (-3.9)	0.5823(-18.5)
	Bim[16]	0.9544	0.8470	0.7419	0.7150
	HPM	0.8587 (-10.5)	0.6418 (-25.5)	0.4868 (-35.6)	0.3852(-44.8)
0	MTS	0.9623 (0.2)	0.8644 (0.2)	0.7391 (-2.3)	0.6144 (-11.9)
	Bim[16]	0.9597	0.8619	0.7570	0.6979
	HPM	0.8990 (-7.55)	0.7158 (-20.5)	0.5637 (-30.4)	0.4552 (-37.4)
0.5	MTS	0.9745 (0.2)	0.9053 (0.4)	0.8095 (-0.1)	0.7050 (3.06)
	Bim[16]	0.9725	0.9011	0.8110	0.7273
	HPM	0.9214 (-5.89)	0.7639 (-17.2)	0.6190 (-26.9)	0.5084 (-33.8)
1	MTS	0.9808 (0.1)	0.9273 (0.4)	0.8500 (0.3)	0.7612 (-0.9)
	Bim[16]	0.9791	0.9230	0.8473	0.7685
	HPM	0.9455 (-4.09)	0.8232 (-13.0)	0.6945 (-21.97)	0.5859 (-29.0)
2	MTS	0.9871 (0.1)	0.9503 (0.3)	0.8947 (0.8)	0.8270 (0.1)
	Bim[16]	0.9859	0.9467	0.8901	0.8254

Table 3. Comparing linear to nonlinear frequency ratio (ω_L / ω_{NL}) for thin four-layered cross-ply laminated plate,(0/90)$_s$and (0/90)$_2$ *a/b*=1, $E_1/E_2 = 25, G_{12}/E_2 = 0.5, \nu_{12} = 0.25, \quad a/h = 75$.

		(0/90)$_s$				**(0/90)$_2$**			
N/N_c^b	Methods	0.25	0.5	0.75	1	0.25	0.5	0.75	1
-0.5	MTS	0.8908	0.6711	0.4755	0.3378	0.8835	0.6547	0.4574	0.3216
	HPM	0.8968	0.7118	0.5597	0.4518	0.8895	0.6975	0.5443	0.4375
-0.125	MTS	0.9349	0.7823	0.6149	0.4732	0.9304	0.7696	0.5975	0.4550
	HPM	0.9370	0.8015	0.6663	0.5566	0.9326	0.7907	0.6525	0.5424
0	MTS	0.9427	0.8043	0.6462	0.5068	0.9386	0.7926	0.6294	0.4885
	HPM	0.9442	0.8201	0.6907	0.5823	0.9403	0.8101	0.6774	0.5683
0.5	MTS	0.9611	0.8607	0.7331	0.6071	0.9583	0.8517	0.7186	0.5895
	HPM	0.9617	0.8689	0.7602	0.6595	0.9591	0.8612	0.7487	0.6462
1	MTS	0.9706	0.8919	0.7857	0.6735	0.9684	0.8846	0.7732	0.6572
	HPM	0.9709	0.8968	0.8037	0.7117	0.9689	0.8905	0.7937	0.6994
2	MTS	0.9802	0.9253	0.8463	0.7560	0.9788	0.9201	0.8366	0.7422
	HPM	0.9803	0.9276	0.8558	0.7786	0.9790	0.9230	0.8479	0.7680

Table 4. Comparison of linear to nonlinear frequency ratio (ω_L/ω_{NL}) for thin four-layered cross ply laminated plate, $(0/90)_s$ and $(0/90)_2$ a/b=1, $E_1/E_2 = 25, G_{12}/E_2 = 0.5, \nu_{12} = 0.25,$ $N/N_c^b = 1$, n=m=1.

		$(0/90)_s$				$(0/90)_2$			
a/h	Methods	**0.25**	**0.5**	**0.75**	**1**	**0.25**	**0.5**	**0.75**	**1**
200	MTS	0.9683	0.8841	0.7722	0.6560	0.9683	0.8843	0.7726	0.6565
	HPM	0.9688	0.8901	0.7930	0.6986	0.9688	0.8903	0.7933	0.6989
100	MTS	0.9703	0.8909	0.7839	0.6711	0.9683	0.8843	0.7725	0.6564
	HPM	0.9707	0.8962	0.8028	0.7105	0.9688	0.8902	0.7932	0.6988
20	MTS	0.9867	0.9487	0.8915	0.8221	0.9683	0.8843	0.7725	0.6564
	HPM	0.9867	0.9498	0.8964	0.8348	0.9688	0.8900	0.7928	0.6981
10	MTS	0.9683	0.8843	0.7725	0.6564	0.9683	0.8843	0.7725	0.6564
	HPM	0.9687	0.8898	0.7922	0.6972	0.9687	0.8898	0.7922	0.6972

Effects of different initial amplitudes and length to thickness ratios on the frequency ratios are investigated in Table 4. According to the results, the influence of thickness of the symmetric and non-symmetric cross-ply composite plates on the frequency ratios is negligible. For high initial amplitudes, by increasing the initial amplitude, the discrepancies of HPM and MTS are increased. For instance, for *a/h=200,* by increasing the initial amplitudes from 0.25 to 1, the discrepancies between HPM and MTS are 0.05% and 6.49% , respectively, for the symmetric plates and 0.05% and 6.45% for the non-symmetric ones. In the case of four-layered plates, the effect of layup (symmetric and non-symmetric) on the frequency ratios is not significant for different length to thickness ratios.Some graphical results are shown to investigate the effects of different initial stresses, initial amplitudes, and aspect ratios on the frequency ratios.

In Fig. 2, the variation of frequency ratio versus initial stress parameter (N/N_c^b) is shown for MTS and HPM frequency analysis methods. Also, the results are compared with those of Bhimaraddi [16]. Variation of nonlinear to linear frequency ratios versus different initial amplitudes for an isotropic plate without initial stresses are investigated by both HPM and MTS and validated using the results of Chen [10] in Fig. 3. According to Figs. 2 and 3, MTS shows more accurate results, especially at high initial amplitudes rather than HPM as compared to Chen's [10] and Bimaraddi's [16] results. According to Fig. 3, both analytical solutions show increasing trends by increasing normal initial stresses, which is analogous to the results by Chen [10] that are obtained using Runge-Kutta solution method.

Variations of linear to nonlinear frequency ratios versus initial amplitude for two special compressive and tensile biaxial initial stresses are investigated in Fig. 4. In both cases, by increasing the initial amplitudes, the discrepancies between two methods are increased.

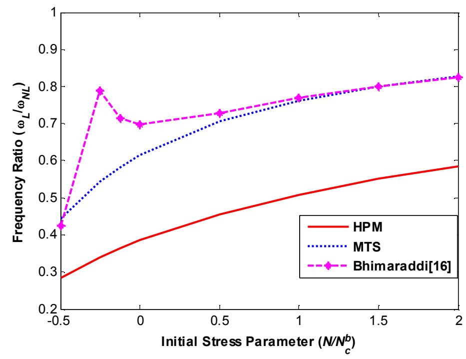

Fig. 2. Frequency ratio versus initial stress parameters are validated with the results of Bhimaraddi[16] for two-layered cross-ply plate (0,90), *W/h*=1, *a/b*=1, *a/h*=10.

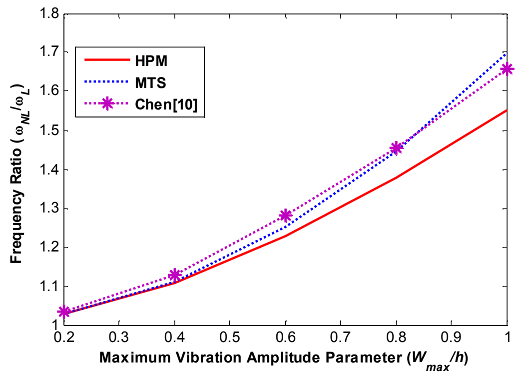

Fig. 3. Frequency ratio versus different initial amplitudes are validated with the results of Chen[10] for an isotropic plate with no initial stresses, *a/b*=1, *a/h*=10, N/N_c^b=1.

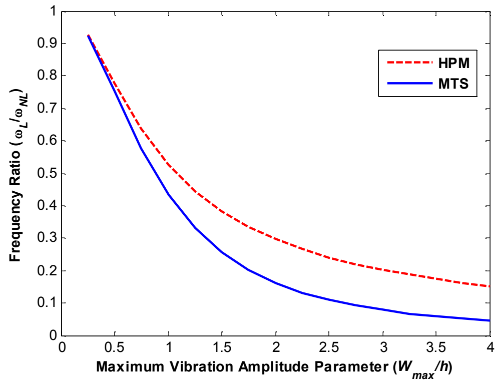

Fig. 4. Frequency ratio versus different initial amplitude for a four-layered symmetric cross-ply plate (0,90,90,0), *a/b*=1, N/N_c^b=-0.25, *a/h*=75.

5. Conclusions

In this work, the nonlinear vibrations of the symmetric and non-symmetric cross-ply composite plates initially stressed by biaxial stresses were investigated. Both homotopy perturbation method and multiple scales perturbation technique as analytical solutions were applied and the numerical results were compared to those available in the literature. Investigating the effects of different initial biaxial stresses, initial amplitudes, and plates' thickness ratio on the frequency ratio indicated that MTS was more adequate than HPM in a wide range of initial amplitude and initial stresses. However, HPM had less computational formulation than MTS. Some new interesting results were presented which have not been previously reported in the literature.

Appendix A[16]

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$
(A2)

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$
(A3)

$$\bar{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta + \cos^4\theta\right)$$
(A4)

Appendix B

$$\alpha = \frac{\left(T_8 + \frac{2T_1T_2T_5 - T_3T_5^2 - T_6T_1^2}{T_3T_6 - T_2^2} + N_x^i\left(\frac{m^2\pi^2}{a^2}\right) + N_y^i\left(\frac{n^2\pi^2}{b^2}\right)\right)}{\left(\sum_{i=1}^{N}\rho_i h_i\right)} \quad \text{(B1)}$$

$$\beta = \frac{\left(T_9 + 3\frac{T_2T_4T_5 + T_1T_2T_7 - T_3T_5T_7 - T_1T_4T_6}{T_3T_6 - T_2^2}\right)}{\left(\sum_{i=1}^{N}\rho_i h_i\right)} \quad \text{(B2)}$$

$$\gamma = \frac{\left(T_{10} + 2\frac{2T_2T_4T_7 - T_3T_7^2 - T_6T_4^2}{T_3T_6 - T_2^2}\right)}{\left(\sum_{i=1}^{N}\rho_i h_i\right)} \quad \text{(B3)}$$

$$T_1 = -\left(\frac{n\pi}{b}\right)^3 B_{22} \quad \text{(B4)}$$

$$T_2 = \left(\frac{n\pi}{b}\right)\left(\frac{m\pi}{a}\right)(A_{12} + A_{66}) \quad \text{(B5)}$$

$$T_3 = \left(\frac{m\pi}{a}\right)^2 A_{66} + \left(\frac{n\pi}{b}\right)^2 A_{22} \quad \text{(B6)}$$

$$T_4 = -\frac{4}{9mn\pi^2}\left[\left(\frac{n\pi}{b}\right)^3 A_{22} + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)(A_{12} - A_{66})\right] \quad \text{(B7)}$$

$$T_5 = -\left(\frac{m\pi}{a}\right)^3 B_{11} \quad \text{(B8)}$$

$$T_6 = \left(\frac{m\pi}{a}\right)^2 A_{11} + \left(\frac{n\pi}{b}\right)^2 A_{66} \quad \text{(B9)}$$

$$T_7 = -\frac{4}{9mn\pi^2}\left[\left(\frac{n\pi}{b}\right)^3 A_{11} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 (A_{12} - A_{66})\right] \quad \text{(B10)}$$

$$T_8 = \left(\frac{m\pi}{a}\right)^4 D_{11} + 2\left[\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 (D_{12} + 2D_{66})\right] + \left(\frac{n\pi}{b}\right)^4 D_{22} \quad \text{(B11)}$$

$$T_9 = -\frac{4(1-(-1)^m)(1-(-1)^n)}{9mn\pi^2}\left[\left(\frac{m\pi}{a}\right)^3 B_{11} + \left(\frac{n\pi}{b}\right)^2 B_{22}\right] \quad \text{(B12)}$$

$$T_{10} = \frac{9}{32}\left[\left(\frac{m\pi}{a}\right)^4 A_{11} + \left(\frac{n\pi}{b}\right)^4 A_{22}\right] +$$

$$\frac{1}{16}\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 (A_{12} + 2A_{66}) \quad \text{(B13)}$$

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} \bar{Q}_{ij}(1, z, z^2)dz, i = j = 1, 2, 6 \quad \text{(B14)}$$

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