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A study on the use of perturbation technique for analyzing the nonlinear forced response of piezoelectric microcantilevers

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Abstract

In this paper, a comparison is made between direct and indirect perturbation approaches to solve the non-linear vibration equations of a piezoelectrically actuated cantilever microbeam. In this comparison, the equation of motion is considered according to Euler-Bernoulli theory with considering the non-linear geometric and inertia terms resulted from shortening effect. In the direct perturbation approach, the multiple scales method is directly applied to the partial differential equation of motion. In the indirect approach, the multiple scales perturbation technique is applied to the discretized equation of motion. It is shown that, if the equation of motion is discretized using one non-uniform microbeam mode shape as a comparison function, then the results of indirect perturbation approach will be identical to those of the direct perturbation approach. Moreover, it is observed that discretization using one uniform microbeam mode shape as a comparison function results in a different output. The concept of non-uniform microbeam mode shape is the linear mode shape of the microbeam by considering the geometric and inertia effects of the piezoelectric layer.

1. Introduction

Piezoelectric effect was discovered by Pierre and Jacob Curry in 1880. Applying electric potential to the material with this characteristic leads to the emergence of strain [1]. Therefore, if this material is deposited on a microbeam and electrical voltage is applied between two sides of piezomaterial, it leads to the curve of the microbeam. Today, this function is used to build micro-actuators or micro-sensors and numerous works have been done to model the dynamic behavior of these systems. Considering that the governing equations are complicated due to the non-linear nature of piezoelectric materials, non-linarites caused by distortion, and non-uniformity of cross-sections, perturbation theory, which is a powerful tool for solving non-linear equations, is used in many studies. Below, the most important background in this regard will be presented. Mahmoodi and Jalili (2007) extracted the non-linear equations governing the vibrations of cantilever micro-beam under AC piezoelectric actuation with considering shortening effect. They used the first mode shape of uniform cantilever beam as a comparison function for discretizing the

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equation of motion and multiple scales method of perturbation theory for analyzing the primary resonance of the separated problem [2]. The concept of uniform microbeam mode shape is the linear mode shape of the microbeam without piezoelectric layer. Similar equations for the piezoelecrically actuated microcantilever to which a bio-layer was added were numerically solved by Mahmoodi et al. in 2007 [3]. In 2008, Mahmoodi and Jalili solved discretized equation of motion for coupled flexural and torsional vibrations of the piezoelectrically actuated microcantilever beam [4]. In 2010, they solved the equations derived from discretization which were extracted from [1] in the secondary resonance mode using multiple scales perturbation theory [5]. In 2012, Shooshtari et al. used one mode shape of uniform microbeam to discretize the equations governing the free vibration of a piezoelectrically actuated microcantilever made of viscoelastic material and, then, solved the obtained conventional differential equation by applying multiple scales method [6]. Hoseini et al. applied multiple scales perturbation method to the forced vibration of the considered model in [7, 8] at primary resonance. In 2013, Korayem and Ghaderi analyzed the vibrational response of a cantilever microbeam with piezoelectric actuation which was exposed to the interaction between the base surface and tip mass attached to the free end of the microbeam. They analyzed the motion equation in a mode shape and solved the obtained results using multiple scales method [9]. The advantage of work [7] over the previous ones was that a non-uniform microbeam mode shape was used as a comparison function in Galerkin method to separate the equations. However, compared with the previous works, the non-linear effects of shortening were neglected. The concept of non-uniform microbeam mode shape was the linear mode shape of the microbeam by considering the geometric and inertia effects of piezoelectric layer. They showed a considerable difference between frequency response curves resulting from applying multiple scales method to a vibration equation separated by a uniform or non-uniform microbeam mode shape as the comparison function. They showed that use of the non-uniform microeam mode shape was in good agreement with the experimental results. This advantage, i.e using non-uniform microbeam mode shape as a comparison function, was used in 2006 by Dick et al. to approximate the vibration of a clamped-clamped microbeam under piezoelectric actuation [10]. In this study, the reason for the formation of non-linearity was middle layer stretching related to the clamped boundary conditions.

In all of the above-mentioned studies, method of perturbation theory is applied to the equations obtained from discretization and, in all of them, only one mode shape is used for discretization. Such an indirect use of perturbation theory is conventional. In some works, perturbation theory method is applied to partial differential equations directly and without separation, which is called direct perturbation theory. Considering non-linearity stemming from shortening assumption, Raeisifard et al. studied the vibrations of cantilever microbeam under piezoelectric actuation using this approach [11]. Lee et al. [12] and Zamanian et al. [13] have used this approach to study the vibrations of clamped-clamped microbeam under piezoelectric actuation considering the non-linearity stemming from middle layer stretching. Recently, the direct perturbation approach was applied by MsCarty and Mahmoodi to the equation of motion of an AFM piezoelectric probe by considering non-linear contact force and neglecting the shortening effect [14, 15].

The literature review shows that, when the indirect perturbation approach has been used in order to consider the non-linear effect of shortening effect, then the equations of motion have been discretized only using one uniform microbeam mode shape as a comparison function. The question that is raised here is that how many modes of uniform microbeam mode shape are needed to give convergence for the solution in this condition. Another question is that what is the effect of using non-uniform microbeam mode shape as the comparison function? The answer can be found by comparing these results with those obtained from solving equations without discretization using direct approach of perturbation theory. Therefore, in this paper, the motion equation is first solved directly and without separation by applying multiple scales perturbation theory. Then, the equations are solved using uniform and non-uniform mode shapes of discretization and multiple scales method. Afterwards, the frequency response functions are compared. These curves must be plotted according to the detuning parameters, i.e. the difference between the excitation frequency and the natural frequency resulted from the related approach. It is noted that, if instead of detuning parameter, they are plotted according to the excitation frequency, then the observed difference would be large due to the computational error of the evaluation of the resonance frequency [16].

2. Direct perturbation approach

A view of the investigated system is shown in Fig. 1. If the dot mark represents the derivative with respect to time, t, and prime symbol denotes the derivative with respect to longitudinal coordinate, s, then the equations of motion and the corresponding boundary conditions for cantilever microbeam will be as the following equation [2]:

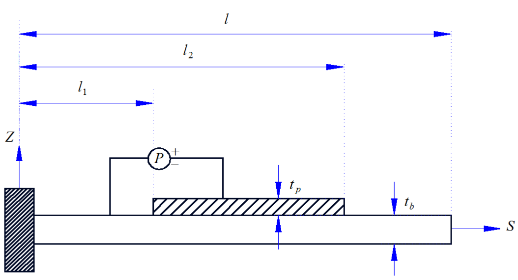

Fig. 1. Schematic view of the microbeam system.

$$m(s)\ddot{v}^{*}+c_E\dot{v}^{*}+\left(C_\eta(s)v^{*\prime\prime}\right)^{\prime\prime}+\underline{\left(v^{*\prime}\left(C_\eta(s)v^{*\prime}v^{*\prime\prime}\right)^{\prime}\right)^{\prime}}$$
$$+\underline{\underline{\left(v^{*\prime}\int_1^s m(s)\left(\int_0^s\left(\dot{v}^{*\prime 2}+v^{*\prime}\ddot{v}^{*\prime}\right)ds\right)ds\right)^{\prime}}}$$
$$+\left(C_\gamma(s)P(t)\right)^{\prime\prime}+\underline{\underline{\underline{\left(C_\gamma(s)P(t)\frac{1}{2}v^{*\prime 2}\right)^{\prime\prime}}}}=0, \tag{1}$$

Where coefficient c_E is the coefficient of viscous damping, and other coefficients i.e. $m(s)$, $C_\eta(s)$, $C_\alpha(s)$ and $C_\gamma(s)$ are expressed as follows:

$$C_\eta(s)=E_bI_bH_\eta(s),\ \ C_\gamma(s)=\frac{E_pd_{31}\bar{A}_P}{t_p}H_\gamma(s),$$
$$H_\gamma(s)=\left(H_{l_1}-H_{l_2}\right),\ H_\eta\left(s\right)=\left(1-H_{l_1}\right)+$$
$$\left(H_{l_1}-H_{l_2}\right)\frac{\bar{I}_b}{I_b}+\left(H_{l_1}-H_{l_2}\right)\frac{E_pI_p}{E_bI_b}+H_{l_2},$$
$$m(s)=w_b\left(\rho_bt_b+\left(H_{l_1}-H_{l_2}\right)\rho_pt_p\right)$$
$$A_b=w_bt_b,\ \mathrm{A}_p=w_pt_p,\ I_b=\frac{w_bt_b{}^3}{12}, \tag{2}$$
$$\bar{I}_b=\frac{w_bt_b{}^3}{12}+t_bw_b\bar{z}_n{}^2,\ \bar{A}_p=\frac{w_p}{2}\left(t_p{}^2+t_pt_b-2t_p\bar{z}_n\right)$$

Where H is the Heaviside function.

$$\mathrm{H}_{\mathrm{l_i}}=\text{Heaviside function }(\mathrm{s}-\mathrm{l_i})=\begin{cases}1 & \mathrm{s}\geq \mathrm{l_i}\\ 0 & \mathrm{s}<\mathrm{l_i}\end{cases} \tag{3}$$
$$\mathrm{i}=1,2$$

Moreover w_b is the width of the microbeam and piezoelectric actuator. ρ_b and ρ_p are the specific densities of the microbeam and piezoelectric actuator, respectively. t_b and t_p are the microbeam and piezoelectric layer thickness, respectively.

In the Eq. (1), the statement with a line drawn under is the nonlinear term caused by bending curvature. Also, the statement that is specified with two lines under is the nonlinear term caused by the inertia. These terms are from the third order and have been appeared for considering the effect of shortening in the equation. In addition, the non-linear term with three lines drawn under has entered the equation due to the piezoelectric layer and is from the second order. This term also has been affected by the effect of shortening. In order to simplify the equation, it is necessary that the equation and the boundary conditions written dimensionless. The following variables are used to do this.

$$v=\frac{v^*}{t_b},\ \ x=\frac{s}{l},\ \ t=\frac{\tau}{T},\ \ T=\sqrt{\frac{\rho_bw_bt_bl^4}{E_bI_b}} \tag{4}$$

Now, in the non-dimensional form, the equation of motion and the associated boundary conditions can be stated as:

$$m(x)\frac{\partial^2 v}{\partial t^2}+C\frac{\partial v}{\partial t}+\frac{\partial^2}{\partial x^2}\left(H_\eta(x)\frac{\partial^2 v}{\partial x^2}\right)+$$
$$\left(\frac{t_b}{l}\right)^2\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(H_\eta(x)\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right)\right)+\left(\frac{t_b}{l}\right)^2\frac{\partial}{\partial x}$$
$$\left(\frac{\partial v}{\partial x}\int_1^x m(x)\left(\int_0^x\frac{\partial^2}{\partial t^2}\left(\frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2\right)dx\right)dx\right)+$$
$$\alpha P(t)\frac{d^2}{dx^2}H_\gamma(x)$$
$$+\alpha P(t)\frac{1}{2}\left(\frac{t_b}{l}\right)^2\times\frac{\partial}{\partial x}\left(\frac{d}{dx}H_\gamma(x)\left(\frac{\partial v}{\partial x}\right)^2\right)=0 \quad (5)$$
$$v_1\big|_{x=0}=0,\ \frac{\partial v_1}{\partial x}\bigg|_{x=0}=0,\ \frac{\partial^2 v_1}{\partial x^2}\bigg|_{x=l}=0,\ \frac{\partial^3 v_1}{\partial x^3}\bigg|_{x=1}=0,$$

where

$$H_\eta(x)=(H_0-H_{l_1/l})+\frac{E^b I_2^b}{E_b I_b}(H_{l_1/l}-H_{l_2/l})+$$
$$\frac{E^p I_2^p}{E_b I_b}(H_{l_1/l}-H_{l_2/l})+(H_{l_2/l}-H_1)$$
$$m(x)=\left(1+\left(H_{l_1/l}-H_{l_2/l}\right)\frac{\rho_p t_p}{\rho_b t_b}\right),$$
$$H_\gamma(x)=(H_{l_1/l}-H_{l_2/l}) \quad (6)$$
$$\alpha=\frac{E_p d_{31}\bar{A}_p l^2}{E_b I_b t_p t_b},\quad C=c_E\sqrt{\frac{l^4}{\rho_b w_b E_b I_b t_b}}$$

Now, in order to apply the multiple scales method, the answer for Eq. (5) is considered:

$$v(x,t)=\varepsilon v_1(x,T_0,T_1,T_2)+$$
$$\varepsilon^2 v_2(x,T_0,T_1,T_2)+\varepsilon^3 v_3(x,T_0,T_1,T_2)+\ldots \quad (7)$$

where ε is an organized dimensionless parameter and represents the order of terms. Also, $T_2=\varepsilon^2 t$,$T_1=\varepsilon t$,$T_0=t$ are different time scales. In this paper, the response is studied in the primary resonance condition. In order to balance the non-linear terms with the terms of viscous damping C and excitation $P(t)$, they are considered order ε^2 and ε^3 terms, respectively. Using the rule of chain differentiation, if Eq. (7) is embedded in differential Eq. (5) and the terms with identical ε exponents equal each other, the equations are obtained at different orders:

order(ε^1):

$$m(x)\frac{\partial^2 v_1}{\partial T_0^2}+\frac{\partial^2}{\partial x^2}\left(H_\eta(x)\frac{\partial^2 v_1}{\partial x^2}\right)=0,$$
$$v_1\big|_{x=0}=0,\ \frac{\partial v_1}{\partial x}\bigg|_{x=0}=0,\ \frac{\partial^2 v_1}{\partial x^2}\bigg|_{x=l}=0,\ \frac{\partial^3 v_1}{\partial x^3}\bigg|_{x=1}=0 \quad (8)$$

order(ε^2):

$$m(x)\frac{\partial^2 v_2}{\partial T_0^2}+\frac{\partial^2}{\partial x^2}\left(H_\eta(x)\frac{\partial^2 v_2}{\partial x^2}\right)=-2m(x)\frac{\partial^2 v_1}{\partial T_1\partial T_0},$$
$$v_2\big|_{x=0}=0,\ \frac{\partial v_2}{\partial x}\bigg|_{x=0}=0,\ \frac{\partial^2 v_2}{\partial x^2}\bigg|_{x=l}=0,\ \frac{\partial^3 v_2}{\partial x^3}\bigg|_{x=1}=0 \quad (9)$$

order(ε^3):

$$m(x)\frac{\partial^2 v_3}{\partial T_0^2}+\frac{\partial^2}{\partial x^2}\left(H_\eta(x)\frac{\partial^2 v_3}{\partial x^2}\right)=$$
$$-m(x)\frac{\partial^2 v_1}{\partial T_1^2}-2m(x)\frac{\partial^2 v_2}{\partial T_1\partial T_0}-2m(x)\frac{\partial^2 v_1}{\partial T_2\partial T_0}-$$
$$C\frac{\partial v_1}{\partial T_0}-\left(\frac{t_b}{l}\right)^2\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}\frac{\partial}{\partial x}\left(H_\eta(x)\frac{\partial v_1}{\partial x}\frac{\partial^2 v_1}{\partial x^2}\right)\right)-$$
$$\left(\frac{t_b}{l}\right)^2\times\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}\int_1^x m(x)\left(\int_0^x\frac{\partial^2}{\partial T_0^2}\left(\frac{1}{2}\left(\frac{\partial v_1}{\partial x}\right)^2\right)dx\right)dx\right)$$
$$-\alpha P(T_0)\frac{d^2}{dx^2}H_\gamma(x),$$
$$v_3\big|_{x=0}=0,\ \frac{\partial v_3}{\partial x}\bigg|_{x=0}=0,\ \frac{\partial^2 v_3}{\partial x^2}\bigg|_{x=l}=0,\ \frac{\partial^3 v_3}{\partial x^3}\bigg|_{x=1}=0 \quad (10)$$

It is observed that Eq. (8) is in fact the linear equation of the system free vibration. Therefore, the solution is as follows:

$$v_1(x,T_0,T_1,T_2)=A(T_1,T_2)e^{I\omega T_0}\Phi(x)+$$
$$\bar{A}(T_1,T_2)e^{-I\omega T_0}\Phi(x), \quad (11)$$

$A(T_1,T_2)$ is a complex function which is achieved by applying solubility conditions at higher orders. Parameter I shows the imaginary part of a complex number. Also, the purpose of $\bar{A}(T_1,T_2)$ is the complex conjugate of $A(T_1,T_2)$ function. $\Phi(x)$ is the mode shape of system linear vibration which is achieved by embedding the modal response as $v(x,t)=\Phi(x)e^{i\omega t}$ in Eq. (8) as follows:

$$-\omega^2 m(x)\Phi(x)+\frac{d^2}{dx^2}\left(H_\eta(x)\frac{d^2\Phi}{dx^2}\right)=0 \quad (12)$$

To solve the above equation, the system is divided into three parts: the first part from the clamped side to the beginning of piezoelectric layer, the second part is the location in which piezoelectric layer is positioned, and the third part is from the end of piezoelectric layer to the free end of microbeam. Therefore, Eq. (12) can be rewritten as Eq. (13):

$$\begin{cases} -\omega^2\Phi(x)+\dfrac{d^4\Phi(x)}{dx^4}=0 & 0\le x<\dfrac{l_1}{l}, \\ -\left(1+\dfrac{\rho_p t_p}{\rho_b t_b}\right)\omega^2\Phi(x)+ & \\ \left(\dfrac{\bar{I}_b}{I_b}+\dfrac{E_p I_p}{E_b I_b}\right)\dfrac{d^4\Phi(x)}{dx^4}=0 & \dfrac{l_1}{l}<x<\dfrac{l_2}{l}, \\ -\omega^2\Phi(x)+\dfrac{d^4\Phi(x)}{dx^4}=0 & \dfrac{l_2}{l}<x\le 1 \end{cases} \tag{13}$$

Solving Eq. (13) in each part is as follows:

$$\begin{cases} \Phi(x)=C_1\cosh(\beta_1 x)+C_2\sinh(\beta_1 x)+ \\ \quad C_3\cos(\beta_1 x)+C_4\sin(\beta_1 x), \quad 0\le x<\dfrac{l_1}{l}, \\ \Phi(x)=C_5\cosh(\beta_2 x)+C_6\sinh(\beta_2 x)+ \\ \quad C_7\cos(\beta_2 x)+C_8\sin(\beta_2 x), \quad \dfrac{l_1}{l}<x<\dfrac{l_2}{l}, \\ \Phi(x)=C_9\cosh(\beta_3 x)+C_{10}\sinh(\beta_3 x)+ \\ \quad C_{11}\cos(\beta_3 x)+C_{12}\sin(\beta_3 x), \quad \dfrac{l_2}{l}<x\le 1 \end{cases} \tag{14}$$

where coefficients C_i are unknown which are obtained by applying the boundary and continuous conditions. By substituting Eq. (11) for Eq. (9), we have:

$$m(x)\frac{\partial^2 v_2}{\partial T_0^2}+\frac{\partial^2}{\partial x^2}\left(H_\eta(x)\frac{\partial^2 v_2}{\partial x^2}\right)=-2m(x)i\omega\frac{\partial A}{\partial T_1}e^{i\omega T_0}\Phi(x)+cc \tag{15}$$

where cc represents the complex conjugate terms available in the equation. Considering that the homogeneous part of the above equation has a non-trivial answer, the equation finds an answer when the solubility conditions are satisfied [12]. Here, the solubility condition is satisfied by zeroing the coefficient of secular terms of Eq. (15). Therefore:

$$\frac{\partial A}{\partial T_1}=0 \ \Rightarrow \ A(T_1,T_2)=A(T_2), \ v_2=0 \tag{16}$$

Since all of the non-homogeneous terms in Eq. (16) are secular and there is no non-secular term in the equation, the private response of this equation is zero. Now, to investigate the main resonance in the third order equation, the AC voltage applied to the piezoelectric layer is considered as shown below:

$$P(T_0)=\frac{1}{2}P_{AC}\left(e^{I\Omega T_0}+e^{-I\Omega T_0}\right),$$
$$\Omega=\omega+\varepsilon^2\sigma \tag{17}$$

where σ is the detuning parameter which shows the difference between linear natural frequency and actuating frequency. By embedding v_1, v_2 from Eq. (11) and (16) in Eq. (10), the following equation is achieved:

$$m(x)\frac{\partial^2 v_3}{\partial T_0^2}+\frac{\partial^2}{\partial x^2}\left(H_\eta(x)\frac{\partial^2 v_3}{\partial x^2}\right)=$$
$$\left(\frac{dA}{dT_2}\hat{m}(x)+A\hat{\mu}+A^2\bar{A}\chi(x)+\hat{F}(x)e^{i\sigma T_2}\right)e^{I\omega T_0}+cc+NST \tag{18}$$

Where NST stands for non secular terms. In addition, the other terms used in the above equation are as follows:

$$\hat{m}(x)=-2m(x)I\omega\Phi(x), \quad \hat{\mu}(x)=-I\omega C\Phi(x), \quad \chi(x)=\chi^G+\chi^E$$
$$\chi^G=-3\left(\frac{t_b}{l}\right)^2\frac{d}{dx}\left(\frac{d\Phi(x)}{dx}\frac{d}{dx}\left(H_\eta(x)\frac{d\Phi(x)}{dx}\frac{d^2\Phi(x)}{dx^2}\right)\right),$$
$$\chi^E=2\omega^2\left(\frac{t_b}{l}\right)^2\frac{d}{dx}\left(\frac{d\Phi(x)}{dx}\int_1^x m(x)\left(\int_0^x\left(\frac{d\Phi(x)}{dx}\right)^2dx\right)dx\right), \tag{19}$$
$$\hat{F}(x)=-\frac{1}{2}\alpha P_{AC}\frac{d^2}{dx^2}H_\gamma(x)$$

For solubility of Eq. (18), the right side of the equation (non-homogeneous part) should be perpendicular on solving the adjoint equation of its homogeneous part [17]. Considering that the homogeneous part of Eq. (18) is self adjoint, therefore, solving the adjoint equation will be similar to the homogeneous response of Eq. (8). Thus, by multiplying the right side of Eq. (19) by $\Phi(x)e^{I\omega T_0}$ and its integration in the interval

$x=0..1$, the solubility conditions will be as follows:

$$2i\omega\left(\tilde{m}\frac{dA}{dT_2}+\frac{\mu A}{2}\right)+8SA^2\bar{A}+Fe^{I\sigma T_2}=0 \tag{20}$$

and the coefficients $\bar{m}$, μ are defined as follows:

$$\tilde{m}=\int_0^1 m(x)\left(\Phi(x)\right)^2 dx=1,\quad \mu=C\int_0^1\left(\Phi(x)\right)^2 dx,$$
$$F=-\int_0^1\hat{F}(x)dx=\frac{1}{2}\alpha P_{AC}\int_0^1\left(\Phi(x)\frac{d^2}{dx^2}H_\gamma(x)\right)dx= \tag{21}$$
$$-\frac{1}{2}\alpha P_{AC}\left(\left.\frac{d\Phi(x)}{dx}\right|_{\frac{l_1}{l}}-\left.\frac{d\Phi(x)}{dx}\right|_{\frac{l_2}{l}}\right)$$

Moreover coefficient S is described as:

$$S=S^G+S^E,$$
$$S^E=-\frac{1}{8}\int_0^1\chi^E\Phi(x)dx=\frac{1}{4}\omega^2\left(\frac{t_b}{l}\right)^2\times$$
$$\int_0^1\left(\left(\frac{d\Phi(x)}{dx}\right)^2\int_1^x m(x)\left(\int_0^x\left(\frac{d\Phi(x)}{dx}\right)^2 dx\right)dx\right)dx$$
$$S^G=-\frac{1}{8}\int_0^1\chi^G\Phi(x)dx=\frac{3}{4}\left(\frac{t_b}{l}\right)^2\times \tag{22}$$
$$\int_0^1 H_\eta(x)\left(\frac{d\Phi(x)}{dx}\frac{d^2\Phi(x)}{dx^2}\right)^2 dx$$

Now, $A(T_2)$ is shown as a polar shape with amplitude $a/2$ and phase β as $A=ae^{I\beta}/2$. By embedding the polar shape A in Eq. (20) and separation of real and imaginary parts, the following is obtained:

$$\frac{da}{dT_2}=-\frac{\mu}{2}a+\frac{F}{\omega}\sin(\gamma)`$$
$$\frac{d\gamma}{dT_2}=\sigma-\frac{S}{\omega}a^2-\frac{F}{a\omega}\cos(\gamma) \tag{23}$$

where $\gamma=\sigma T_2-\beta$. In order to achieve the steady state response of the system, the time derivatives in Eq. (23) should be equal to zero. Therefore, the relationship between a and σ will be:

$$a^2\left(\left(\frac{\mu}{2}\right)^2+\left(\sigma-\frac{S}{\omega}a^2\right)^2\right)=\left(\frac{F}{\omega}\right)^2 \tag{24}$$

According to the above equation, the maximum amplitude (which occurs in resonance condition) is achieved by equaling the values of statement inside the third parenthesis to zero. Therefore, the amount of a and σ in the resonance will be as follows:

$$a=\frac{2F}{\omega\mu},\quad \sigma=\frac{S}{\omega}a^2 \tag{25}$$

By embedding $\Omega=\omega+\varepsilon^2\sigma$ and $\varepsilon=1$ in the above equation, the resonance frequency is obtained:

$$\Omega=\omega+\frac{4SF^2}{\omega^3\mu^2} \tag{26}$$

Finally, by applying Eqs. (11) and (16) and also $\varepsilon=1$ in the Eq. (7), the system response will be as:

$$v(x,t)=a\cos(\Omega t-\gamma)\Phi(x) \tag{27}$$

3. Indirect perturbation approach

3. 1. Discretization

In this state, before the application of the multiple scales method, partial differential Eq. (5) should be discretized. The Galerkin method is used for discretization. Therefore, the response of equation is considered as follows:

$$v(x,t)=\sum_{i=1}^{M}q_i(t)\Phi_i(x) \tag{28}$$

where $\Phi_i(x)$ is the comparison function that should satisfy the dynamic and geometric boundary conditions and $q_i(t), i=1..M$ are time coordinates that are achieved using multiple scales method. Also, M is the number of mode shape. Embedding Eq. (28) in the motion Eq. (5), multiplying the result by $\Phi_j(x),\ j=1..M$, and integrating it in the interval $x=0..1$ lead to a set of separated equations as:

$$eq_j=\sum_{i=1}^{M}m_j^i\frac{d^2q_i}{dt^2}+\sum_{i=1}^{M}k_j^iq_i+\sum_{m,n,p=1}^{M}g_{1j}^{mnp}q_mq_nq_p+\sum_{m,n,p=1}^{M}g_{2j}^{mnp}\left(q_m\frac{dq_n}{dt}\frac{dq_p}{dt}+q_mq_n\frac{d^2q_p}{dt^2}\right)+\sum_{i=1}^{M}\mu_j^i\frac{dq_i}{dt}$$
$$g_{3j}P(t)+P(t)\sum_{m,n=1}^{M}g_{4j}^{mn}q_mq_n=0,,\quad j=1..M \tag{29}$$

The parameters used in Eq. (29) are defined as below:

$$m_j^i=\int_0^1 m(x)\Phi_j\Phi_i dx,$$
$$\mu_j^i=C\int_0^1\Phi_j\Phi_i dx,\ k_j^i=\int_0^1\frac{d^2\Phi_j}{dx^2}H_\eta(x)\frac{d^2\Phi_i}{dx^2}dx,$$
$$g_{1j}^{mnp}=\left(\frac{t_b}{l}\right)^2\int_0^1\left(\begin{array}{c}\frac{d^2\Phi_j}{d^2x}H_\eta(x)\\ \frac{d\Phi_m}{dx}\frac{d\Phi_n}{dx}\frac{d^2\Phi_p}{d^2x}\end{array}\right)dx+\left(\frac{t_b}{l}\right)^2\int_0^1\left(\begin{array}{c}\frac{d\Phi_j}{dx}H_\eta(x)\\ \frac{d^2\Phi_m}{d^2x}\frac{d\Phi_n}{dx}\frac{d^2\Phi_p}{d^2x}\end{array}\right)dx,$$
$$g_{2j}^{mnp}=\left(\frac{t_b}{l}\right)^2\int_0^1\left(\begin{array}{c}\frac{d\Phi_j}{dx}\frac{d\Phi_m}{dx}\\ \int_1^x m(x)\left(\int_0^x\frac{d\Phi_n}{dx}\frac{d\Phi_p}{dx}dx\right)dx\end{array}\right)dx,$$
$$g_{3j}=\alpha\int_0^1\left(\Phi_j\frac{d^2}{dx^2}H_\gamma(x)\right)dx=-\alpha\left(\left.\frac{d\Phi_j}{dx}\right|_{x=\frac{l_1}{l}}-\left.\frac{d\Phi_j}{dx}\right|_{x=\frac{l_2}{l}}\right),$$
$$g_{4j}^{mn}=\frac{1}{2}\alpha\left(\frac{t_b}{l}\right)^2\left(\begin{array}{c}\left.\frac{d\Phi_j}{dx}\frac{d\Phi_m}{dx}\frac{d\Phi_n}{dx}\right|_{x=\frac{l_1}{l}}\\ -\left.\frac{d\Phi_j}{dx}\frac{d\Phi_m}{dx}\frac{d\Phi_n}{dx}\right|_{x=\frac{l_2}{l}}\end{array}\right) \tag{30}$$

3. 2. Using non-uniform microbeam mode shape

If one non-uniform microbeam mode shape, i.e. the solution of Eq. (13), is used as the comparison function for discretization, then Eqs. (29) and (30) are summarized as Eqs. (31) and (32):

$$\frac{d^2q}{dt^2}+\mu\frac{dq}{dt}+\omega^2q+g_2\left(q\left(\frac{dq}{dt}\right)^2+q^2\frac{d^2q}{dt^2}\right)+g_1q^3+g_3P(t)+g_4P(t)q^2=0 \tag{31}$$

where:

$$\mu=C\int_0^1\Phi^2dx,\ g_1=2\left(\frac{t_b}{l}\right)^2\int_0^1H_\eta(x)\left(\frac{d\Phi}{dx}\frac{d^2\Phi}{dx^2}\right)^2dx,$$
$$g_4=\frac{1}{2}\alpha\left(\frac{t_b}{l}\right)^2\left(\left.\left(\frac{d\Phi}{dx}\right)^3\right|_{\frac{l_1}{l}}-\left.\left(\frac{d\Phi}{dx}\right)^3\right|_{\frac{l_2}{l}}\right),$$
$$g_2=-\int_0^1\left(\left(\frac{d\Phi}{dx}\right)^2\int_1^x m(x)\left(\int_0^x\left(\frac{d\Phi}{dx}\right)^2dx\right)dx\right)dx,$$
$$g_3=-\alpha\left(\left.\frac{d\Phi}{dx}\right|_{\frac{l_1}{l}}-\left.\frac{d\Phi}{dx}\right|_{\frac{l_2}{l}}\right) \tag{32}$$

Now, for applying multiple scales method (by considering the fact that the second order non-linear term does not participate in the system response), the response of Eq. (31) is assumed as follows:

$$q(t)=\varepsilon q_1(T_0,T_2)+\varepsilon^3q_3(T_0,T_2) \tag{33}$$

where ε is an organized dimensionless parameter and also $T_2=\varepsilon^2t$, $T_0=t$ are different time scales. In order to balance the damping and excitation terms with nonlinear terms, their orders are considered equal to ε^2 and ε^3, respectively. Now, if Eq. (33) is embedded in the differential Eq. (31) and the terms with identical ε exponents equal each other, the equations obtained at different orders will be as:

order(ε):

$$\frac{\partial^2q_1}{\partial T_0^{\,2}}+\omega^2q_1=0 \tag{34}$$

order(ε^3):

$$\frac{\partial^2q_3}{\partial T_0^{\,2}}+\omega^2q_3=-2\frac{\partial^2q_1}{\partial T_2\partial T_0}-\mu\frac{\partial q_1}{\partial T_0}-g_1q_1^{\,3}-g_2\left(q_1\left(\frac{\partial q_1}{\partial T_0}\right)^2+q_1^{\,2}\frac{\partial^2q_1}{\partial T_0^{\,2}}\right)-g_3P(T_0) \tag{35}$$

Response of Eq. (34) is as follows:

$$q_1(T_0,T_2)=A(T_2)e^{I\omega T_0}+\bar{A}(T_2)e^{-I\omega T_0} \tag{36}$$

Now, to investigate the main resonance in the third order equation, the applied voltage to the piezoelectric layer is considered similar to Eq. (17). By embedding Eqs. (17) and (36) in Eq. (35), the following is obtained:

$$\frac{\partial^2 q_3}{\partial T_0^{\,2}}+\omega^2 q_3=(-2I\omega\frac{dA}{dT_2}-I\omega\mu A-3g_1A^2\bar{A}+ 2\omega^2 g_2A^2\bar{A}-\frac{1}{2}g_3P_{AC}e^{I\sigma T_2})e^{I\omega T_0} +cc+NST \tag{37}$$

In order to determine the solubility conditions of the above equation, the secular terms should be eliminated, which will be as the following equation by equalizing $e^{I\omega T_0}$ coefficients to zero:

$$-2I\omega\frac{dA}{dT_2}-I\omega\mu A-3g_1A^2\bar{A}+ 2\omega^2 g_2A^2\bar{A}-\frac{1}{2}g_3P_{AC}e^{I\sigma T_2}=0 \tag{38}$$

3. 3. Using uniform microbeam mode shape

In this section, the uniform microbeam mode shape or the simple microbeam mode shape without the piezoelectric layer that have been used as a comparative function in major previous studies [2-6] is used for discretization. In contrast to previous works that have used one mode shape as the comparison function, here, several modes are used. So, the set of ordinary differential equation described in Eq. (29) must be solved. For applying the multiple scales method, the response of Eq. (29) is assumed as follows:

$$q_i(t)=\varepsilon q_{i1}(T_0,T_2)+\varepsilon^3 q_{i3}(T_0,T_2),\quad i=1..M \tag{39}$$

However, if Eq. (39) is embedded in differential Eq. (29) and the terms with identical ε exponents equal each other, the equations are obtained at different orders:

order(ε):

$$\sum_{i=1}^{M} m_j^i\frac{d^2 q_{i1}}{dT_0^{\,2}}+\sum_{i=1}^{M}k_j^i q_{i1}=0,\quad j=1..M \tag{40}$$

Order(ε^3):

$$\sum_{i=1}^{M} m_j^i\frac{\partial^2 q_{i3}}{\partial T_0^{\,2}}+\sum_{i=1}^{M}k_j^i q_{i3}= -2\sum_{i=1}^{M} m_j^i\frac{\partial^2 q_{i1}}{\partial T_2\partial T_0}-\sum_{i=1}^{M}\mu_j^i\frac{\partial q_{j1}}{\partial T_0} -\sum_{m,n,p=1}^{M} g_{1j}^{mnp}q_{m1}q_{n1}q_{p1}+g_{3j}P(T_0)- \sum_{m,n,p=1}^{M} g_{2j}^{mnp}\left(q_{m1}\frac{\partial q_{n1}}{\partial T_0}\frac{\partial q_{p1}}{\partial T_0}+ q_{m1}q_{n1}\frac{\partial^2 q_{p1}}{\partial T_0^{\,2}}\right),\quad j=1..M \tag{41}$$

Eq. (40) includes a set of the coupled differential equations. Hence, their responses are considered as follows:

$$q_{i1}(T_0,T_2)=c_i e^{I\omega T_0},\quad \mathrm{i}=1..M \tag{42}$$

Embedding Eq. (42) in (40) leads to a set of equations based on c_j . Zeroing the determinant of the matrix coefficients of these equations leads to the deterministic equation. By solving this equation, a set of frequencies is obtained. Therefore, the response of Eq. (40) will be in accordance with Eq. (43):

$$q_{j1}(T_0,T_2)=\sum_{i=1}^{M}(\varsigma_i^j A_i(T_2)e^{I\omega_i T_0}+\bar{\varsigma}_i^j\bar{A}_i(T_2)e^{-I\omega_i T_0}),\quad j=1..M\ \&\ \varsigma_i^1=1 \tag{43}$$

Where, ς_i^j represents the relation between c_j s , and is from the matrix of coefficients. Now, to investigate the main resonance in the third order equation, the applied voltage to the piezoelectric layer is considered in accordance with Eq. (17). By embedding the Eq. (43) in (34), Eq. (44) is achieved.

$$\begin{aligned}&\sum_{i=1}^{M} m_j^i \frac{\partial^2 q_{i3}}{\partial {T_0}^2} + \sum_{i=1}^{M} k_j^i q_{i3} = \\ &\sum_{k,i=1}^{M}\left(\begin{aligned}&-2I\omega_k\left(\sum_{r=1}^{M} m_j^r \varsigma_k^r\right)\frac{dA_k}{dT_2} - \\ &\sum_{k,i=1}^{M} I\omega_k \sum_{r=1}^{M} \mu_j^r \varsigma_k^r A_k - \\ &\chi_{ik}^j A_i \bar{A}_i A_k - \frac{1}{2} g_{3j} P_{AC} e^{i\sigma T_2} \delta_{1k}\end{aligned}\right) e^{I\omega_k T_0} \\ &+cc + NST, \; j = 1..M\end{aligned} \tag{44}$$

In the above equation, the parameter χ_{ik}^j is defined as follows:

$$\begin{aligned}&\delta_{1k} = \begin{cases}1 & \text{if} \quad k = 1, \\ 0 & \text{if} \quad k \neq 1,\end{cases} \\ &\chi_{ik}^j = \sum_{m,n,p=1}^{M}\Big(g_{1j}^{mnp}\big(\bar{\varsigma}_i^m \varsigma_i^n \varsigma_k^p + \bar{\varsigma}_i^m \varsigma_k^n \varsigma_i^p + \\ &\varsigma_i^m \bar{\varsigma}_i^n \varsigma_k^p + \varsigma_k^m \bar{\varsigma}_i^n \varsigma_i^p + \varsigma_i^m \varsigma_k^n \bar{\varsigma}_i^p + \varsigma_k^m \varsigma_i^n \bar{\varsigma}_i^p \\ &-\bar{\varsigma}_k^m \varsigma_k^n \varsigma_k^p - \varsigma_k^m \bar{\varsigma}_k^n \varsigma_k^p - \varsigma_k^m \varsigma_k^n \bar{\varsigma}_k^p\big)\Big) + g_{2j}^{mnp} \\ &(-\omega_i \omega_k \bar{\varsigma}_i^m \varsigma_i^n \varsigma_k^p - \omega_i \omega_k \bar{\varsigma}_i^m \varsigma_k^n \varsigma_i^p \\ &-\omega_i \omega_k \varsigma_i^m \bar{\varsigma}_i^n \varsigma_k^p + \omega_i \omega_k \varsigma_i^m \varsigma_k^n \bar{\varsigma}_i^p - \\ &{\omega_k}^2 \bar{\varsigma}_i^m \varsigma_i^n \varsigma_k^p - {\omega_i}^2 \bar{\varsigma}_i^m \varsigma_k^n \varsigma_i^p {\omega_k}^2 \varsigma_i^m \bar{\varsigma}_i^n \varsigma_k^p \\ &-{\omega_i}^2 \varsigma_i^m \varsigma_k^n \bar{\varsigma}_i^p + 2{\omega_k}^2 \bar{\varsigma}_k^m \varsigma_k^n \varsigma_k^p)\end{aligned} \tag{45}$$

The private solution of Eq. (44) can be considered as follows:

$$q_{j3}(T_0, T_2) = \sum_{i=1}^{M} Q_i^j(T_2) e^{I\omega_i T_0}, \qquad j = 1..M \tag{46}$$

Now, the Eq. (46) is embedded in Eq. (44) and the coefficients of each $e^{I\omega_k T_0}$ exponent at both sides of the equalization have been equaled. Considering that the homogeneous part of the equations has non-trivial answer, the non-homogeneous equations only will have non-trivial response if the determinant of the matrix resulted from elimination of a column of the adding matrix equaled to zero. Therefore, the solubility conditions will be in accordance with Eq. (47):

$$\begin{vmatrix}\left(k_1^1 - {\omega_k}^2 m_1^1\right) & \cdots & \left(k_1^{M-1} - {\omega_k}^2 m_1^{M-1}\right) & \Gamma_1^k \\ \vdots & \ddots & \vdots & \vdots \\ \left(k_M^1 - {\omega_k}^2 m_M^1\right) & \cdots & \left(k_M^{M-1} - {\omega_k}^2 m_M^{M-1}\right) & \Gamma_M^k\end{vmatrix}, \quad k = 1..M \tag{47}$$

where Γ_j^k is the coefficient of $e^{I\omega_k T_0}$ in the $j\,th$ equation from the set of Eqs. (43). After the calculation of the solubility conditions of the above equation, the amplitude $A_k(T_2)$ is applied in its polar shape to the equations; then, with its calculation in the steady state, the response of discretized motion Eq. (29) is obtained. Comparison of the calculated amplitude in this part with the response of direct multiple scales method will lead to an appropriate opinion of accuracy and convergence of the response caused by uniform microbeam mode shape as a comparative function in the discretization of motion equations.

4. Results and discussion

In the calculations, the geometric and mechanical properties mentioned in Table 1 are used along with C=0.0005. Figures 2 and 3 show that, if the equation of motion is discretized using a non-uniform microbeam mode shape as the comparison function, then applying the multiple scales perturbation method to the discretized motion equation leads to the similar frequency response to the direct application of perturbation method to the partial differential equation. In Fig. 2, one end of the piezoelectric layer is positioned on the fixed side of the micro-cantilever beam. In Fig. 3, the piezoelectric layer is separated from the fixed support and deposited in the middle part of the micro-beam. It means that, in this condition, the agreement between frequency responses resulting from direct and indirect approach is not depended on the piezoelectric layer position.

Afterwards, the convergence of response in the indirect method in the case of using uniform microbeam mode shape for the discretization of equations used in many previous studies [2-6] was examined and compared with the direct solution method.

Figure 4 shows the frequency response of the microcantilever in the case that the one end of the piezoelectric layer is positioned on the fixed side of the microbeam. The results show that, in this case, if the uniform microbeam mode shape is used as a comparative function, at least three mode shapes should be used for better assessment of the effect of non-linear terms in the vibrating equation.

In Fig. 5, the piezoelectric layer is separated from the fixed support and deposited in the middle part of the microbeam. It is observed that, in this case, there is an egregious error in the calculated response (both amplitude and non-linear effects) using a uniform mode shape. Therefore, if the uniform micro-beam mode shape is used as a comparative function for the discretization of partial differential equations, to achieve the response with sufficient accuracy, it is necessary to use several mode shapes.

Table 1. Mechanical and geometric properties used in the analysis of system response.

L (m)	E_b (Gpa)	(Gpa)	ρ_b (kgm^{-3})
200	160	67	2300
ρ_p (kgm^{-3})	d_{31} ($volt^{-1}$)		(volt)
7700	0.000175	0.5	0.002

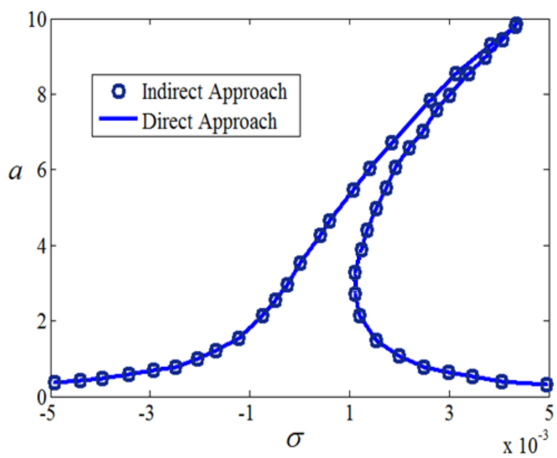

Fig. 2. Comparison between direct and indirect perturbation approaches where the indirect approach is applied to the discretized equation using non-uniform microbeam mode shape as the comparison function , $l_1 = 0$, $l_2 = 0.5l$.

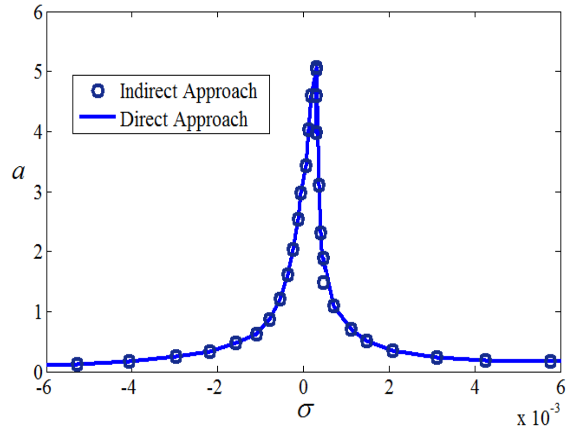

Fig. 3. Comparison between direct and indirect perturbation approaches where the indirect approach is applied to the discretized equation using non-uniform microbeam mode shape as the comparison function , $l_2 = 0.75l$, $l_1 = 0.25l$.

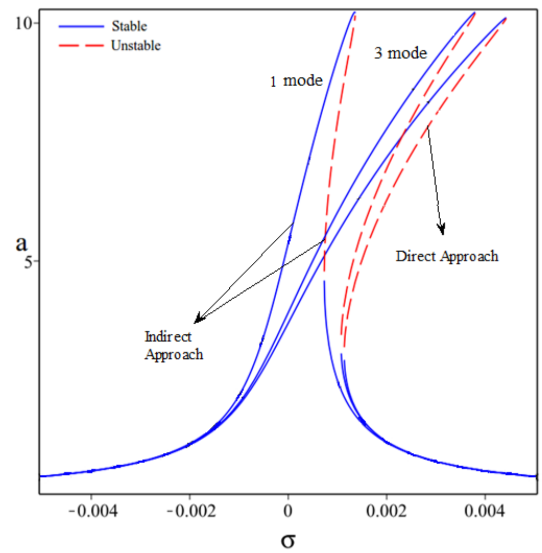

Fig. 4. Comparison between direct and indirect perturbation approaches where the indirect approach is applied to the discretized equation using uniform microbeam mode shape as the comparison function, $l_1 = 0$, $l_2 = 0.5l$.

This consequence represents the fact that the results of previous research, in which one uniform microbeam mode shape has been used as the comparison function, are not sufficiently accurate. Although by increasing the number of mode shapes, the accuracy increases, it needs solving a set of complicated non-linear equations. The results demonstrate that, for modifying the results, it is preferred to apply perturbation theory directly to the partial differential equations or apply perturbation theory to the discretized equation using non-uniform microbeam mode shape as the comparison function.

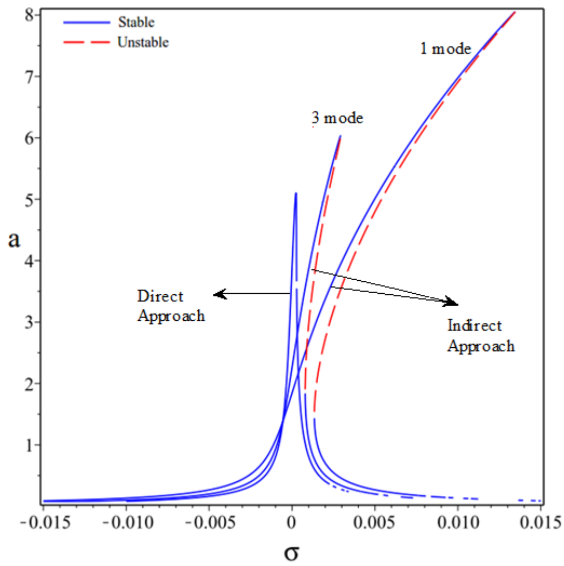

Fig. 5. Comparison between direct and indirect perturbation approaches where the indirect approach is applied to the discretized equation using uniform microbeam mode shape as the comparison function , $l_2 = 0.75l$, $l_1 = 0.25l$.

5. Conclusions

In this paper, a comparison was made between direct and indirect perturbation approaches for solving the non-linear vibration equations of a piezoelectrically actuated cantilever microbeam. In this comparison, the equation of motion was considered according to Euler-Bernoulli theory with considering the non-linear geometric and inertia terms resulting from shortening effect. In the direct perturbation approach, the multiple scales method was directly applied to the partial differential equation of motion. In the indirect approach, the multiple scales perturbation technique was applied to the discretized equations of motion. It was shown that, if the equation of motion were discretized using one non-uniform microbeam mode shape as the comparison function, then the results of indirect perturbation approach would be identical to those of the direct perturbation approach. It was also observed that, if one end of the piezoelectric layer were positioned on the clamped side of the microbeam, and the equation of motion were discretized using uniform microbeam mode shape as the comparison function, then at least three mode shapes would be used to provide an agreement between direct and indirect perturbation techniques. Moreover, the error rate increased with the separation of piezoelectric layer from the clamped side of the microbeam.

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