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Research paper

Evaluation of improved precise integration method in structural dynamics using single value decomposition technique

J. Akbari[a,*] and S. Mirzaei[b]

[a] *Department of Civil Engineering, Bu-Ali Sina University, Hamedan, Iran*
[b] *Genova University, Savona compose, Genova, Italy*

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***Corresponding author:**
j.akbari@basu.ac.ir and jalal.akbari@gmail.com

Abstract

Numerical techniques for solving dynamic structural problems often encounter significant challenges, including conditional stability, period elongation errors, amplitude decay errors, and the emergence of spurious frequencies. To address these issues, several first-order precise integration methods have been developed; however, these approaches still suffer from errors associated with the inversion of the state matrix. This study employs the singular value decomposition technique to enhance the efficiency of the precise integration method algorithm and eliminate the singularity of the state matrix. The robustness of the proposed method is evaluated across various transient dynamic problems. The results demonstrate that traditional approaches, such as the Newmark method, exhibit substantially larger errors—exceeding 150% in certain cases. Ultimately, the findings emphasize that accurately estimating the dynamic response of multi-degree-of-freedom systems under impact loading requires careful consideration. Conventional methods, including the Newmark average acceleration technique, should therefore not be applied indiscriminately.

1. Introduction

The evaluation of transient dynamic responses using numerical methods is typically classified into two main categories. The first category comprises methods that directly solve the second-order equilibrium equations, while the second includes algorithms that transform the second-order equations into first-order equations in the state-space form. The primary challenges associated with these methods include conditional stability, period elongation errors, amplitude decay errors, and the emergence of spurious frequencies. In practical applications, second-order methods are widely employed, with certain approaches demonstrating greater popularity due to their computational efficiency and accuracy. Among the second-order methods, the Newmark average acceleration method, despite its

unconditional stability status, has the error of spurious frequency, and it also does not have the capability to eliminate the undesirable effects of the higher modes [1].

The order of the equation has been reduced and converted to a first-order equation, so-called precise integration method (PIM) in the state space, to solve the stability problem in second-order methods. These methods also have stability, accuracy, and state matrix inversion errors. If the matrix is single or ill-conditioned, numerical errors appear. In recent years, comprehensive research has been conducted on precise integration methods.

The PIM method was introduced by Zhang and William [2]. Due to the inverse matrix error and load simplification, the accuracy of this method is reduced when the equation of motion is inhomogeneous.

Guangzhou *et al.* [3] proposed a new version of PIM to convert an inhomogeneous equation to a homogeneous one with a dimensional expansion method. Regardless of computing the inverse matrix, this method is computationally expensive.

Wang and Zhou [4] demonstrated that this method has conditional stability. Wong and Au [5] have proposed the precise time step integration method (PTSIM) to solve the equilibrium equation with Gaussian techniques. The accuracy of the method depends only on the number of Gaussian points and the time step size; however, this method is conditionally stable.

As well, Wong and Au [5] presented the NICPIM method to resolve the inverse matrix. They further reported that this method is unconditionally stable. However, if the state matrix is singular or ill-conditioned, significant numerical errors appear in the calculations. Investigations in first-order PIM methods could be categorized into two-fold: the First area struggles in solving the accuracy and stability problems of the precise integration methods from the mathematical point of view. The second field is in the application of the PIM methods in structural dynamics and time-varying areas.

Many efforts have been made to improve the stability, accuracy, and elimination of the matrix inverse error. Many researchers have investigated the improvement of the PIM method from a mathematical point of view. Fu *et al.* [6] used this method to solve ill-condition algebraic equations; Zhang and Huang [7] in solving singular boundary value problems; Fu and Li [8] to resolve ill-conditioned linear equations. Besides, Goa *et al.*[9] used improved PIM to solve the Riccati differential equation. Finally, Tan *et al.* [10, 11] applied the Fourier series expansion in the improvement of PIM to solve differential equations.

Many kinds of research have been accomplished in the structural dynamic area. Wang [12], Wu and Chuang [13, 14] applied the PIM method for solving the equilibrium equation of motion in structural dynamics. In other disciplines of civil and structural engineering, the PIM method is involved in various time-varying problems.

Gao *et al.* [15, 16], Cai *et al.* [17], Yue *et al.* [18], Ding and Hu [19] used the precise integration method (PIM) and improved PIM for dynamic response and vibration analysis of large structures. Also, a vibration analysis of the footbridge is carried out by Caprani [20]. Modal analysis of ocean risers with PIM is carried out by Zhou *et al*. [21] Hu and Wang [22], applied the PIM for the time-history analysis of frame-shear walls and tube-type structural systems.

The PIM for dynamic load identification by Lingling and Yang [23] and also the Pseudo-dynamic test of structures by Wang and Liu [24] are successfully applied. Besides, the PIM method is used by many researchers, e.g., Han *et al.*[25] for dynamic response of footing on soil; Qingguan [26] in dynamic analysis of gate chamber; Zhang *et al.* [27] to solve dynamic nonlinear behavior of wheel-rail contact; Du *et al.* [28] and Hu *et al.* [29] in the dynamic interaction of train and bridge; Fang *et al.*[30] in the modeling of wave propagation in the layered pavement; Zhang *et al.* [31, 32] in earthquake pounding model and elasto-plastic seismic analysis of adjacent structures.

Wu *et al.* [33] in the modeling of periodic structures; Chen *et al.* [34] for sensitivity analysis in time domain response of transmission lines; Cui *et al.*[35] in the dynamic response of a beam with crack; and finally

This study enhances the first-order PIM method to eliminate ill-conditioning and also inverse matrix errors. For this purpose, the single-value decomposition technique is combined with the

classic PIM method, and the proposed method is called PIMS. The technique has the desirable accuracy and stability to eliminate the matrix inverse error.

The results show that the accuracy of the proposed method is better than the classic PIM and also the second-order Newmark average acceleration methods, which are inherently unconditionally stable. Several structural dynamic problems illustrate the robustness of the proposed method. The capabilities of the proposed PIM and the Newmark average acceleration method are experimentally evaluated. The results show that the estimation of dynamic responses of MDF structural systems under impulse loading required more investigations, and these methods could not be blindly used.

2. Formulation of improved PIM

2.1. Theory of classic PIM

In this section, the theory of the classic precise integration method is presented. Consider a second-order dynamic equilibrium equation of a damped system as Eq. (1).

$$\mathbf{M}\,\ddot{\mathbf{X}}(t)+\mathbf{C}\,\dot{\mathbf{X}}(t)+\mathbf{K}\,\mathbf{X}(t)=\mathbf{F}(t) \tag{1}$$

where **M**, **C,** and **K** are the mass, damping, and stiffness matrices of the system, respectively. **F**(t) is an external force vector. As well, **X,** $\dot{\boldsymbol{X}}$, and $\ddot{\boldsymbol{X}}$ refer to displacement, velocity, and acceleration vectors, respectively. The second-order equation in Eq. (1) is transformed into a first-order equation in the state space as Eq. (2) for obtaining the PIM formulation.

$$\begin{cases}\dot{\mathbf{v}}=\mathbf{A_c}\mathbf{v}+\mathbf{E_c}\mathbf{F}\\ \mathbf{X}=\mathbf{D}\,\mathbf{v}\end{cases},\ \mathbf{A_c}=\begin{bmatrix}\mathbf{o} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}$$
$$\mathbf{E_c}=\begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}\mathbf{I} & \mathbf{o}\end{bmatrix},\ \mathbf{v}=\begin{bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{bmatrix} \tag{2}$$

where **v** is the state variable, $\mathbf{A_c}$ is the state matrix, **D** is the input-output coupling matrix, $\mathbf{E_c}$ is called the input distribution matrix, **I** is the identity matrix, and **F** refers to the controller vector. If $\mathbf{E_c}$ becomes zero, there is no control over external force (**F**), and the problem is time-independent. The discrete form of Eq. (2) in the time domain is written as Eq. (3).

$$\mathbf{v}(t)=\mathbf{e}^{\mathbf{A_c}t}\mathbf{v_0}+\mathbf{e}^{\mathbf{A_c}t}\int_0^t\mathbf{e}^{-\mathbf{A_c}t}\mathbf{E_c}\,\mathbf{F}(s)\,ds \tag{3}$$
$$\mathbf{v_{n+1}}=\mathbf{T}\mathbf{v_n}+\mathbf{E_0}\mathbf{F_n}+\mathbf{E_1}\mathbf{F_{n+1}}$$
$$\mathbf{E_0}=\left(\mathbf{A_c}^{-1}\mathbf{T}+\frac{1}{\Delta t}\mathbf{A_c}^{-2}(\mathbf{I}-\mathbf{T})\right)\mathbf{E_c}\ ,$$
$$\mathbf{E_1}=\left(-\mathbf{A_c}^{-1}+\frac{1}{\Delta t}\mathbf{A_c}^{-2}(\mathbf{T}-\mathbf{I})\right)\mathbf{E_c}$$

In Eq. (3), **F** is linearly interpolated between two sequential time intervals. e^{At} is called the transformation state matrix, and the accuracy of the method depends on the precision of calculating this matrix. **T** matrix in Eq. (3) is written as Eq. (4).

$$\mathbf{T}(\Delta t)=e^{\mathbf{A}_c\Delta t}=\left(e^{\mathbf{A}_c\frac{\Delta t}{m}}\right)^m=\left(e^{\mathbf{A}_c\tau}\right)^m=(\mathbf{T}(\tau))^m \tag{4}$$

In this paper, $m=2^N$ as proposed by Molar and Van [36] and Wu and Chuang [13], suggests N = 5 or 4. T (τ) is calculated by using the Taylor series expansion as Eq. (5).

$$\mathbf{T}(\tau)=\mathbf{I}+\mathbf{T}_{a0}\ ,\ \mathbf{T}_{a0}=\mathbf{A}_c\tau+\frac{(\mathbf{A}_c)^2}{2!}+\frac{(\mathbf{A}_c)^3}{3!}+\ldots\ldots+\frac{(\mathbf{A}_c)^L}{L!} \tag{5}$$

where I refers to an identity matrix and, because only the first terms of the Taylor series expansion have adequate accuracy, the higher terms of the series are ignored. According to the research of Wang and Au [5], the calculation is carried out for L = 4 and is written as Eq. (6).

$$\mathbf{T}(\Delta t)\cong\left(\mathbf{I}+\mathbf{T}_{a0}\right)^{2^N} \tag{6}$$

Based on the recursive relationships, the $\mathbf{T}_{ai}$ matrix is written as Eq. (7).

$$\left(\mathbf{I}+\mathbf{T}_{a(N-1)}\right)^2=\mathbf{I}+2\times\mathbf{T}_{a(N-1)}+$$
$$\mathbf{T}_{a(N-1)}\times\mathbf{T}_{a(N-1)}=\mathbf{I}+\mathbf{T}_{aN} \quad (7)$$
where N=1, n+1

The elements of the matrix $\mathbf{T}_{ai}$ are exceedingly small. To reach better numerical accuracy, instead of directly adding $\mathbf{T}_{ai}$ to the identity matrix, they are stored separately from the identity matrix **I** during the calculations. Thus, the computational accuracy of the **T**(Δt) matrix is increased by avoiding round-off errors during computer calculations.

According to Eq. (3), the calculation of the structural response in the PIM requires the inverse calculation of the state matrix $\mathbf{A}_c$; therefore, when the matrix is singular or ill-conditioned, an error will occur in the calculations. In this study, to avoid this error, the inverse SVD method is implemented, as described in Section 2.2. In the state-space equations, the transfer function expresses the characteristics of any linear system.

In general, the transfer function is defined as the ratio of the Laplace transform of the system output to that of the input, assuming all initial conditions are zero. Accordingly, by applying the Laplace transform to Eq. (2), the transfer function can be derived as expressed in Eq. (8).

$$s\mathbf{v}(s)=\mathbf{A}_c\mathbf{v}(s)+\mathbf{E}_c\mathbf{F}(s)\rightarrow$$
$$(s\mathbf{I}-\mathbf{A}_c)\mathbf{v}(s)=\mathbf{E}_c\mathbf{F}(s)$$
$$\mathbf{X}(s)=\mathbf{C}\mathbf{v}(s):\ \mathbf{X}(s)=\frac{\mathbf{C}\ \mathbf{E}_c\ \mathbf{F}(s)}{(s\mathbf{I}-\mathbf{A}_c)} \quad (8)$$
$$\mathbf{G}(s)=\frac{\mathbf{X}(s)}{\mathbf{F}(s)}=\frac{\mathbf{C}}{(s\mathbf{I}-\mathbf{A}_c)}=$$
$$\frac{\mathbf{C}\ adj\,(s\mathbf{I}-\mathbf{A}_c)}{\det|s\mathbf{I}-\mathbf{A}_c|}=\frac{Y\ (s)}{R\ (s)}$$

where the rational function **G**(s) denotes the transfer function, Y(s) represents the Laplace transform of the output, and R(s) is the Laplace transform of the input. The denominator is called the characteristic equation of the state system. As well, the roots of the characteristic equation are called poles of the transfer function; the time response and the stability of the method depend on the poles of the transfer function. In the PIM, the eigenvalues $\mathbf{A}_c$ are defined as Eq. (9).

$$|s\mathbf{I}-\mathbf{A}_c|=0\rightarrow\begin{vmatrix} s & -\mathbf{I} \\ \mathbf{M}^{-1}\mathbf{K} & s+\mathbf{M}^{-1}\mathbf{C}\end{vmatrix}=0 \quad (9)$$

where s are the eigenvalues of $\mathbf{A}_c$, **M**, and **K** are the mass and stiffness matrices, respectively. For a damped SDF system, Eq. (9) is written as Eq. (10).

$$s^2+2\xi\omega s+\omega^2=0$$
$$s=-\xi\omega\pm\sqrt{\xi^2-1}=-\xi\omega\pm i\sqrt{1-\xi^2} \quad (10)$$

where ω and ξ are the frequency and damping of the system, respectively. The method is stable if the spectral radius ρ($\mathbf{A}_c$) is smaller than one. In the PIM, the eigenvalues of the matrix $\mathbf{A}_c$ are calculated as Eq. (11).

$$\rho_1,\rho_2=\alpha\pm i\beta=$$
$$e^{-\zeta\omega\Delta t}(\cos\bar{\omega}\Delta t\pm i\sin\bar{\omega}\Delta t) \quad (11)$$
$$\rho=\max(|\rho_1|,|\rho_2|)$$

A single degree of freedom of structure under free vibration is considered to investigate the stability of the proposed method. In the system, the mass of the system is 5000 (kg), and the stiffness is set to 3000 (kN/m). Fig. 1 presents the variations of spectral radius against dt/T, in which T is the fundamental period of the system.

According to Fig. 1, this method is stable for N equal to or greater than 4 or 5. As well, Fig. 2 shows that the PIM method is unconditionally stable when dt/T≤0.5 for all N values; However, the values of N are important when dt/T≥0.5, and PIM is unconditionally stable for N≥4.

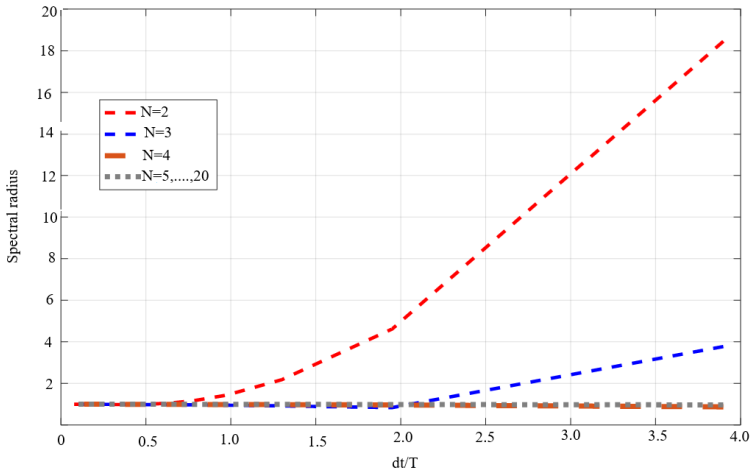

Fig. 1. Variations of spectral radius vs. dt/T for classic PIM.

2.2. Single value decomposition technique (SVD)

One of the problems with the first-order method is calculating the inverse of the state matrix . If this matrix is ill-conditioned or singular, there is a significant error in the numerical computations. In this study, this weakness is resolved by the inverse matrix (SVD), even if the state matrix is singular or ill-conditioned, the calculations will be accurate.

In the SVD, matrix $\mathbf{A}_{mxn}$ can be decomposed into $\mathbf{A} = \mathbf{UDV}^T$. Where $\mathbf{U}_{m\times n} = [u_1 \dots u_m]$ and $\mathbf{V}_{n\times n} = [v_1 \dots v_n]$ are orthogonal matrices whose columns $\mathbf{U}_{mxn}$ are $\mathbf{AA}^T$ matrix eigenvalues, $\mathbf{V}_{n\times n}$ columns are $\mathbf{A}^T\mathbf{A}$ matrix eigenvectors, and according to Eq. (12) $\mathbf{D}_{n\times n}$ is a diagonal matrix whose diagonal elements are non-zero values of the square root of the eigenvalues of both $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ matrices.

$$\begin{aligned} &D_{n\times n} = \mathrm{diag}(\sigma_1,\dots,\sigma_p),\ p=\min\{m,n\} \\ &\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k > 0 \quad ,\ \sigma_{k+1}=\dots\sigma_p=0 \end{aligned} \tag{12}$$

here σ_1 and σ_k are the largest and smallest non-zero singular values of matrix $\mathbf{A}$. For $\mathbf{A}_{nxn}$ square matrix, the inverse matrix is defined as Eq. (13).

$$\begin{aligned} \mathbf{A}^{-1} &= (\mathbf{U}D\mathbf{V}^T)^{-1} = (\mathbf{V}^T)^{-1}D^{-1}\mathbf{U}^{-1} \\ &= \mathbf{V}D^{-1}\mathbf{U}^T \end{aligned} \tag{13}$$

where $D^{-1} = \mathrm{diag}(1/\sigma_1, 1/\sigma_2, \dots, 1/\sigma_n)$. The accuracy of the improved method is also investigated for the structure of a damped single degree of freedom. Here, the mass of a system is 5000 kg, the stiffness is 3000 kN/m, ξ is 0.05, and the harmonic excitation $p(t) = 5000\sin(2\pi t)$ (N). The responses of the PIM without SVD and the PIM with SVD were compared with the analytical response. The equilibrium equation of an SDOF with boundary conditions is defined as Eq. (14)

$$m\ddot{x}+c\dot{x}+kx=p(t)\ ;\ x(0)=0\ ,\ \dot{x}(0)=0 \tag{14}$$

The displacements at times 0.10, 0.20, 0.30, 0.40, and 0.50 s are presented in Table 1. The exact structural response is obtained as Eq. (15) by solving Eq. (14). The time step in numerical analysis is set to 0.05.

$$\begin{aligned} x(t) = \frac{\pi P_o}{2K\xi}((\exp^{(-\xi t)}[\cos(\omega_D t) + \\ \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_D t)] - \cos(\pi t)) \end{aligned} \tag{15}$$

where $\omega_D = \omega\sqrt{1-\xi^2}$. According to Table 1, the singularity or ill-conditioning of the coefficient matrix in the PIM method causes an extreme error in the calculations. The responses of the system with the inverse of SVD are also shown

in Fig. 2. It can be seen that the proposed PIMS method is highly accurate, and the results have excellent agreement with the analytical responses. Hereafter, the PIM method using SVD will be called PIMS.

2.3. Spurious frequencies in the proposed PIM (PIMS)

When analyzing systems with a substantial number of degrees of freedom (DOFs), higher modes that are artificially introduced by the finite element model must be appropriately filtered during direct response calculations. Numerical methods that introduce controlled damping to suppress these higher modes—without significantly influencing the behavior of the lower modes—are generally preferred.
So far, the accuracy of the method has been studied when the equilibrium equations are homogeneous. Few studies have investigated the robustness of the algorithm under external loading. A new numerical study is introduced by Paramount [37] under external excitation. In his method, a separate equation transfer function was introduced. A comparison between the calculated transfer function of the analytical method and the numerical method shows that the transfer function gives comprehensive information for the accuracy of the method and the demonstration of spurious frequencies. The transfer function for f(t)= $\exp(i\,\bar{\omega}_n \Delta t)$ is calculated from Eq. (16) [1].

$$H_x = \frac{1}{\omega^2 - \bar{\omega}_n{}^2 + 2i\xi\omega_n\,\bar{\omega}_n} \tag{16}$$

The relation between the external force and the system response at a particular time is considered as Eq. (17).

$$\begin{bmatrix} \mathrm{x} \\ \dot{\mathrm{x}} \end{bmatrix} = \begin{bmatrix} \mathrm{H}_x \\ \mathrm{H}_{\dot{x}} \end{bmatrix} \exp(i\,\bar{\omega}_n \Delta t) \tag{17}$$

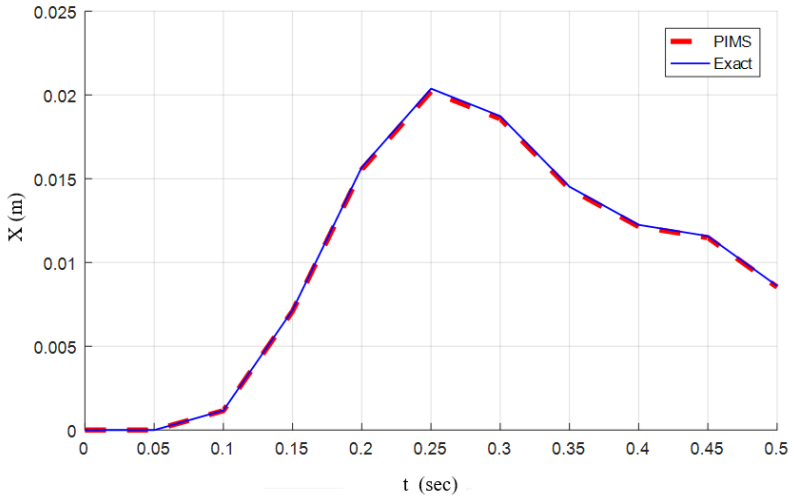

Fig. 2. Free vibration response of the damped system for proposed PIM (PIMS).

Table 1. Comparing the responses of a damped SDF system (units are mm).

Method of calculations	**Time (s)**				
	0.10	0.20	0.30	0.40	**0.50**
Exact	0.00117	0.0157	0.0188	0.0123	**0.0086**
Proposed PIM	0.00116	0.0156	0.0186	0.0121	**0.0085**
Classic PIM	2014	9423	7878	5799	**2333**
Proposed PIM error (%)	0.85	0.64	1.06	1.62	**1.16**
Classic PIM error (%)	1.7e8	6.0e7	4.1e7	4.7e7	**2.7e7**

H_x is the displacement transfer function, and $H_{\dot{x}}$ is the velocity transfer function. By substituting Eq. (17) into Eq. (2), Eq. (18) is obtained.

$$\begin{bmatrix} \bar{H}_x \\ \bar{H}_{\dot{x}} \end{bmatrix} = \left[\exp(i\,\bar{\omega}_n \Delta t)\times \mathbf{I} - \mathbf{T}\right]^{-1} * \left[\mathbf{E}_0 \quad \mathbf{E}_1\right] * \begin{bmatrix} \mathbf{I} \\ \exp(i\,\bar{\omega}_n \Delta t) \end{bmatrix} \tag{18}$$

$\bar{H}_x$ is the PIM transfer function and is calculated by using Eq. (18), where E_0 and E_1 are defined in Eq. (3). When $\left|\bar{H}_x / H_x\right| = 1|$, it means that the numerical values are equal to the analytical ones, and no spurious frequencies are generated in the method. In the presence of external loading, the period elongation and the amplitude decay errors do not indicate the accuracy of the method. The ratio of the analytic transfer function to the numerical transfer function shows the accuracy of the method. The variations of $\left|\bar{\mathbf{H}}_x / \mathbf{H}_x\right|$ against $\bar{\omega}_n / \omega_n$ for the PIMS and average acceleration Newmark methods are presented in Fig. 3. The time step $\Delta t = 0.1\times T$ is considered for the SDF system studied in Section 2.4.

Fig. 3 shows that the proposed PIMs method does not have a spurious frequency, and also, when $\bar{\omega}_n / \omega_n \le 1$, or more precisely, it is less than 0.6, the response of the method is very close to the analytical method. However, in the Newmark average acceleration method, when $\bar{\omega}_n / \omega_n = 1$, the spurious frequency appears, and a discontinuity is seen. However, there are many transfer functions for the appropriate estimation of responses in time and frequency domains [38].

2.4. Filtering high-frequency spurious responses

In addressing dynamic problems, numerical methods often exhibit relative period errors in the representation of higher modes. Consequently, the high-frequency responses predicted by numerical models are likely to be spurious, thereby diminishing the overall accuracy of the numerical response analysis. Therefore, only the first few predominant modes have a significant influence on the responses, and the contribution of the higher-order frequency responses is sometimes spurious and inaccurate.

They reduce the accuracy of the calculations and also cause numerical instability. As a result, these frequencies are not desirable, and they should be filtered. If linear systems are controllable, the internal stability of the system can be guaranteed by designing a controller based on the system's feedback. It reduces model sensitivity and eliminates disturbance effects. In linear time-invariant systems (LTI), this controller is considered a linear function [14].

By calculating the feedback of the system and optimal pole placement, the effect of perturbation and spurious responses is considerably reduced. In the PIM, a matrix is added to the damping matrix (**C**) by calculating the feedback; it is called $\mathbf{C}_a$ [11]. According to Eq. (19), the matrix $\mathbf{C}_a$ is a function of the stiffness matrix **K** and Δt.

$$\mathbf{C}' = \mathbf{C} + \mathbf{C}_a = \mathbf{C} + 2\gamma \Delta t \mathbf{K} \tag{19}$$

According to the feedback of the system, the value of the spectral radius is obtained by Eq. (20).

$$\rho_1, \rho_2 = \begin{cases} \exp(-\xi^* \omega_n \Delta t \pm i\,\omega_n \Delta t \sqrt{1-\xi^{*2}}) & 0 \le \xi^* \le 1 \\ \exp(-\xi^* \omega_n \Delta t \pm i\,\omega_n \Delta t \sqrt{1-\xi^{*2}}) & \xi^* \ge 1 \end{cases} \qquad \rho = \max(|\rho_1|, |\rho_2|) \tag{20}$$

where ω is the natural frequency of the system, the damping coefficient must be changed to the value of ξ^*, as defined by Eq. (21).

$$\xi^* = \frac{C + 2\alpha K \Delta t}{2M\,\omega_n} = \xi + 2\gamma\pi \frac{\Delta t}{T} \tag{21}$$

In the single degree of freedom (SDOF) system, γ is defined as Eq. (22).

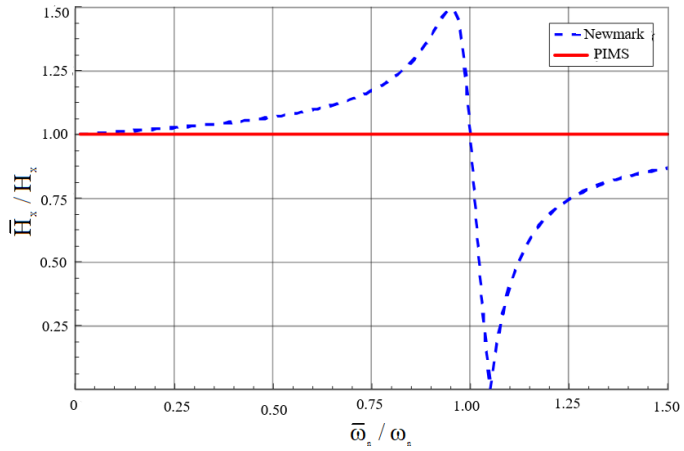

Fig. 3. Spurious frequency variations vs. $\bar{\omega}_n / \omega_n$.

$$\gamma = -\ln\rho / \left(\omega_n \Delta t\right)^2 \quad 0 \le \xi \le 1$$
$$\gamma = \frac{-(1+\left(\frac{\ln\rho}{\omega_n \Delta t}\right)^2)}{2\ln\rho} \quad \xi^* \ge 1 \tag{22}$$

The variation of the spectral radius of the PIM with feedback was also calculated for the mentioned system described in Section 2.1. The spectral radius variations, including feedback control, are illustrated in Fig. 4. According to this figure, the proposed PIM method with feedback is unconditionally stable, and its spectral radius for all values of N is smaller than one. Therefore, the presented method is unconditionally stable due to the feedback controller.

2.5. The accuracy of the proposed method

Similar to the second-order methods, the PIM method also has amplitude decay error (AD) and period elongation error (PE). These were calculated for a single degree of freedom structure under free vibration. The frequency of the structure in this method is calculated according to Eq. (23).

$$\omega^* = \begin{cases} \dfrac{\tan^{-1}(|\beta|/\alpha)}{\Delta t} & \alpha > 0 \\ \dfrac{\tan^{-1}(|\beta|/\alpha) + \pi}{\Delta t} & \alpha < 0 \end{cases} \tag{23}$$

The period elongation error was obtained by comparing the natural frequency of the structure (ω) and the PIM frequency (ω^*) as it is defined in Eq. (24).

$$PE = \frac{\omega}{\omega^*} - 1 \tag{24}$$

In this method, the amplitude decay error (AD) is also calculated as Eq. (25).

$$\xi^* = \frac{-\ln(\alpha^2 + \beta^2)}{2\omega^* \Delta t} \quad , \quad AD = 2\pi\xi^* \tag{25}$$

An undamped single degree of freedom system in Section 2.1 is again considered under free vibration to study the accuracy of the proposed method. The frequency error and the amplitude decay error are shown in Fig. 5. According to Fig. 5, the first-order methods for N = 2 have a period elongation error, and it decreases with increasing N value, and for N = 4, the error is negligible. The error of the method for amplitude decay is also shown in the Figure. The PIM amplitude error is also insignificant for N larger than 4.

3. Results and discussion

In this section, the performance of the proposed PIM (PIMS) is evaluated with several structural dynamics problems. For this purpose, three structural models are examined.

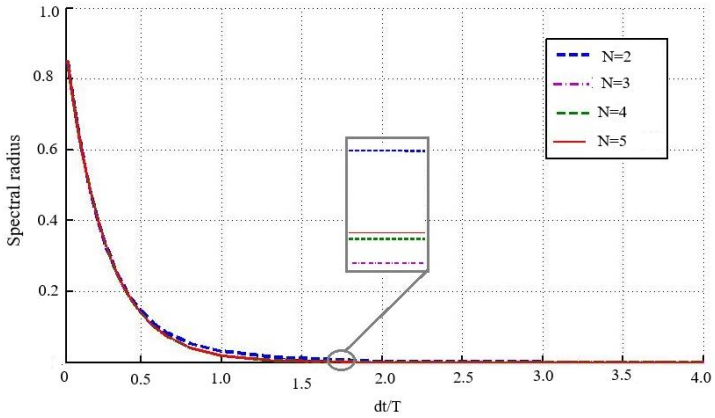

Fig. 4. Variations of spectral radius vs. dt/T including feedback on the system.

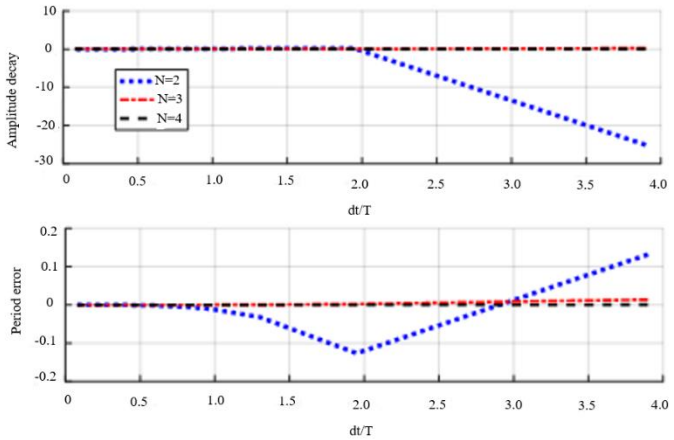

Fig. 5. Amplitude and period elongation errors of the proposed PIM (PIMS).

The first one is a damped SDOF system under the harmonic and rectangular impulse loadings. Its dynamic response was investigated by analytical, PIMS, and second-order Newmark average acceleration methods.

The second case is a two-degree-of-freedom structural model to examine the feasibility of filtering high frequencies. The third case refers to an aluminum cantilever beam with a concentrated mass at the free end using the Euler-Bernoulli theory.

The structural response of this beam was compared under a very short impulse load with experimental data and also the Newmark and PIMS methods. All procedures explained in this paper are programmed in the MATLAB environment [39].

3.1. Case 1: A damped SDF system

The objective is to investigate the dynamic responses of a single degree of freedom (SDOF) structure with the specifications presented in Table 2. The response of this system is evaluated by analytical, Newmark average acceleration, and the PIMS methods (see Table 3). When the load frequency is close to the frequency of the structure, the results of the Newmark method have some errors. As seen in Table 3, it is observed that the response of the PIMS method is close to the analytical solution, whereas the Newmark method results have more substantial errors. The structural response of the mentioned SDF system under impulse-loading with intensity P(t)=5000(kN) with time-duration 0.2 s, and the results of the above-mentioned methods for times 0, 0.2, 0.4, 0.6, 0.8, and 1 s are presented in Table 4. According to Table 4, it can be seen that the PIMS method shows better accuracy under impulse loading.

Table 2. Specification of a damped SDF system (harmonic excitation).

m (kg)	k (kN/m)	ξ	$\bar{\omega}$ (rad/s)	p(t) (kN)
5000	3000	0.05	22.5	$2000\sin(\bar{\omega}t)$

Table 3. Displacements of a damped SDF system under harmonic excitation (mm).

Evaluation method	Time (second)					Maximum Error (%)
	0	1	2	3	4	
Analytical	0	0.0758	-0.0032	-0.0383	0.0321	0
PIMS (This research)	0	0.0738	-0.0031	-0.0373	0.0312	3.1
Newmark	**0**	**0.095**	**-0.0420**	**-0.0300**	**0.0215**	33

Table 4. Displacements of a damped SDF system under impulse loading (mm).

Evaluation method	**Time (s)**						**Maximum error (%)**
	0	0.2	0.4	0.6	0.8	1.0	
Analytical	0	0.0013	0.0186	-0.00066	-0.0021	-0.0011	0
PIMS (This research)	0	0.0013	0.0205	-0.00057	-0.0022	-0.0081	24
Newmark	0	0.1500	0.0168	-0.00102	-0.002	0.0037	44

3.2. Case 2: A two-degree-of-freedom system

The average acceleration Newmark method**,** known for its unconditional stability, is widely recommended for evaluating dynamic responses using the modal superposition technique, where truncation of the modal space inherently filters high-frequency components. This method offers superior accuracy, and the absence of numerical damping does not adversely affect the results. In the present study, the performance and applicability of the average acceleration Newmark method are investigated within the framework of the direct integration approach**,** as an alternative to the conventional modal superposition method, through the analysis of a multi-degree-of-freedom (MDOF) system. The unconditionally stable average acceleration Newmark method is recommended for a calculation of the response by modal superposition when the filtering of the high frequencies is naturally carried out by using a truncated modal space. It is the most accurate method, and the absence of numerical damping is not a problem. In this paper, the capabilities of the average acceleration Newmark method with the direct integration method, instead of the modal superposition method, are investigated by studying an MDF system. For this purpose, consider a freedom undamped structural model as seen in Fig. 6. The first mode is regarded as the desired mode and the second mode as the unwanted high-order mode that must be filtered.

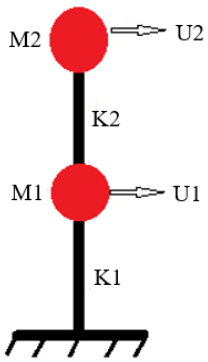

Fig. 6. Schematic figure of two DOF system, $M_1 = 2000\ kg$, $M_2 = 4M_1$, $k_1 = 1800\ kN/m$, and $k_2 = 180000\ kN/m$.

The displacement responses of the first degree of freedom (u_1) in Fig. 7 and the response of the second degree of freedom (u_2) in Fig. 8 are presented with the effect of filtering the high modes. As shown in the figures, the higher mode effect is not eliminated by either the PIMS or Newmark methods. As a result, both methods do not have appropriate capabilities to reduce the high frequencies.

Case 3: aluminum cantilever beam

The purpose of this case is to investigate the abilities of the proposed PIMS and the average acceleration Newmark method for the beam under short-time loading (see Fig. 9). The geometry of the beam and the mechanical and geometrical properties of the beam are presented in Table 5. The history of the exerted hammer load in the free end of the beam is measured in the laboratory and is shown in Fig. 10.

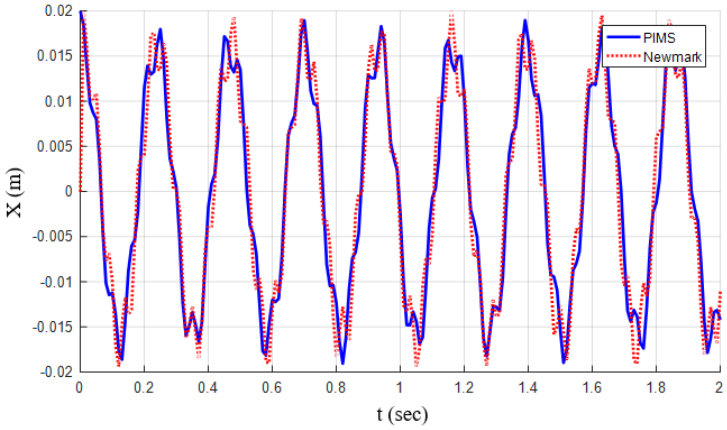

Fig. 7. The u_1 response of the system was evaluated using the PIM and Newmark methods.

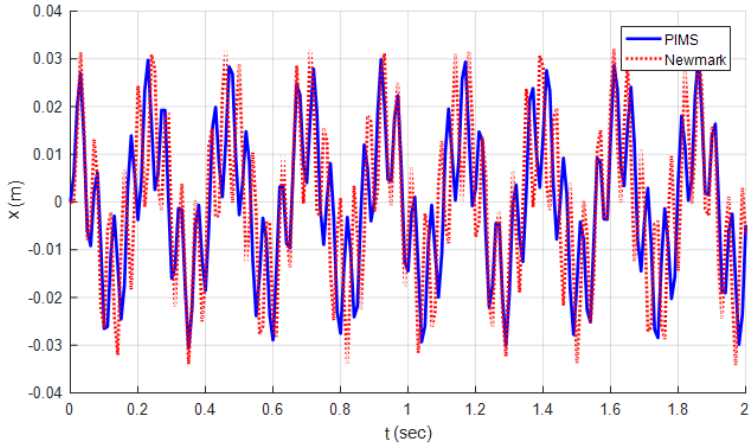

Fig. 8. The u_2 response of the system was analyzed and compared using both the PIM and Newmark methods.

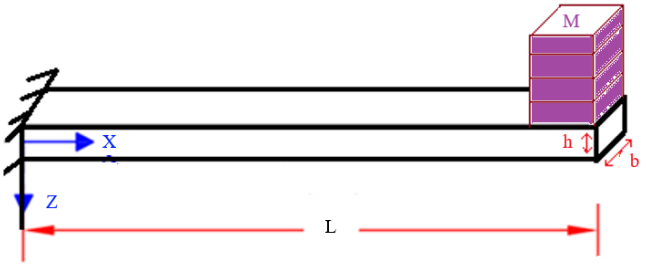

Fig. 9. Aluminium cantilever beam.

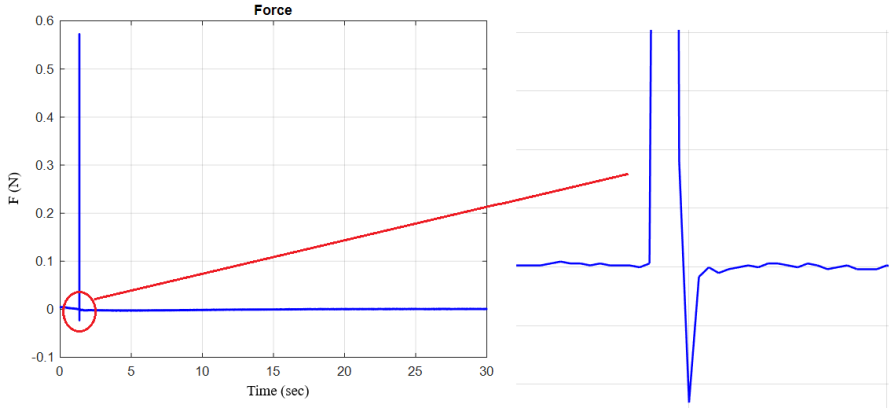

Fig. 10. Experimental data of hammer-induced impact loads at the free end of the beam.

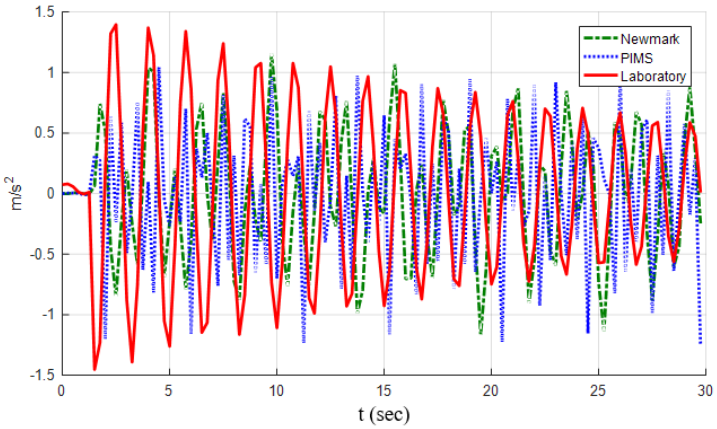

Fig. 11. Acceleration of the free end of the beam with experimental data, Newmark, and PIMS methods (without damping $\xi = 0$).

Table 5. Specifications of an aluminum beam.

ρ (kg/m^3)	B (mm)	h (mm)	L (mm)	$\bar{m}$ (kg/m)	M (kg)	E (Gpa)
2700	40	10	750	1.08	1.585	69

The finite-element model of the beam is created and also updated using FRF data for this study. Therefore, in the PIMS and Newmark average acceleration methods, the updated mass, damping, and mass matrices are used. Damping is even negligible in high-velocity and very short time loadings; therefore, in this research, the results are reported for $\xi = 0,\ 0.1\%$. According to Fig. 11, for the undamped case ($\xi = 0$), in both numerical methods, the amplitudes of vibrations are not appropriately damped. However, because of the low damping ratio in the system, the amplitude of vibration in the experimental results gradually decayed. As seen from Fig. 11, in the first times of vibrations, the numerical results are less than the experimental ones, and from the middle to the end of the vibration times, the numerical results are larger than the experimental response.

For the low ratio of damping case ($\xi = 0.2\%$), in both numerical methods, the amplitudes of vibrations are steadily damped. Also, because of the presence of low damping in the beam, the amplitude of vibration from the experimental results gradually decayed. As seen from Fig. 12, the amplitudes of vibrations in both numerical methods are less than experimental data, and these results show that for finite-element modeling problems under short-time loading, more investigations should be considered in applying the proposed PIM and Newmark average acceleration method.

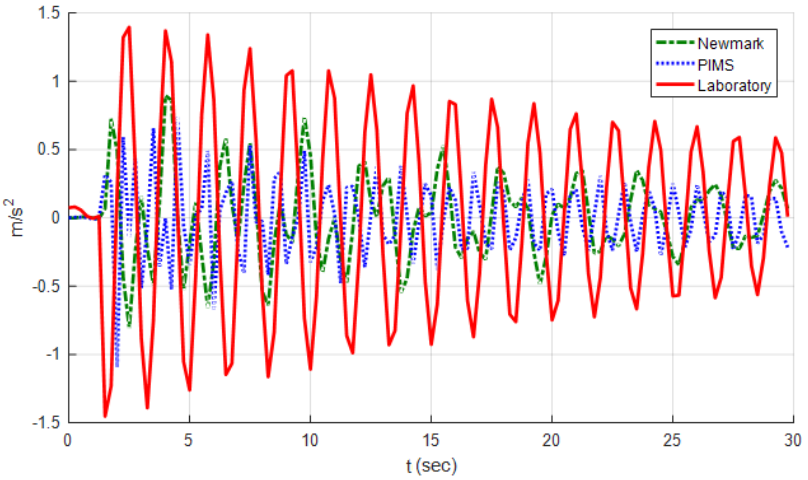

Fig. 12. Acceleration of the free end of the beam with experimental data, Newmark, and PIMS methods with damping ($\xi = 0.2\%$).

4. Conclusions

This study proposes an improved formulation of the Precise Integration Method (PIM) to achieve higher accuracy in the analysis of structural dynamics problems. The improvement has been accomplished by inversion of the state matrix through a singular matrix decomposition (SVD) technique. The stability, ill-conditioning, and spurious frequency errors have been eliminated with the proposed method. Based on the case studies, the following results could be drawn:

- The possible errors during numerical computations due to the singularity of the state matrix have been eliminated using the singular matrix decomposition (SVD) technique.
- The stability of the proposed method is guaranteed for any problems in the state matrix. The PIMS is unconditionally stable for N≥4.
- By adding the feedback to the proposed method, the values of the spectral radius are stable for all N values. Therefore, the PIMS method with feedback has unconditional stability.
- In the modeling of structural systems with large finite-element DOF, and also for models under short-time loading, e.g., impact or shock, well-known techniques, e.g. The Newmark method cannot be used blindly, and dynamic responses of such systems have high errors.
- The well-known Newmark method and this proposed PIM method are not able to eliminate high-frequency effects in MDF systems. Therefore, for modeling structures with the Newmark method, using a modal superposition approach is preferable to the direct integration method.

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Conflict of interests

The authors declare that there are no conflicts of interest associated with this work.

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