



## Approximate solution of laminar thermal boundary layer over a thin plate heated from below by convection

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### Abstract

In this paper, an integration of a symbolic power series method - Padé approximation technique (PS - Padé), was utilized to solve a system of nonlinear differential equations arising from the similarity solution of laminar thermal boundary layer over a flat plate subjected to a convective surface boundary condition. As both boundary conditions tended to infinity, the combination of series solutions with the Padé approximants was used for handling boundary conditions on the semi-infinite domain of solution. The combination of power series and Padé proposed an alternative approach of solution which did not require small parameters and avoided linearization and physically unrealistic assumptions. The results of the present approach were compared with numerical results as well as those of previous works reported in the literature. The obtained results represented remarkable accuracy in comparison with the numerical ones. Finally, reduced Nusselt number, as an important parameter in heat transfer, was calculated by the obtained analytical solution. The present power series-Padé technique was very simple and effective, which could develop a simple analytic solution for flow and heat transfer over the flat plate. The results of the present study could be easily used in practical applications.

### Nomenclature

$A, B$	constant matrix	$Pr$	Prandtl number
$c, a$	a constant	$T$	temperature
$f$	similarity function for stream function	$T_f$	temperature of fluid flow below the plate
$f, y, e$	vector functions	$T_w$	temperature of the plate surface
$h_f$	convection heat transfer coefficient	$T_\infty$	free stream temperature
$i, k, l$	dummy parameter	$u$	velocity component in x-direction
$k$	thermal conductivity	$U_\infty$	free stream velocity
$Nu$	Nusselt number	$\nu$	kinematic viscosity
$Re_x$	local Reynolds number	$X$	dimensional variable, independent variable in the definition of power
$p, q$	vector functions		
$f_w$	transpiration rate at the surface		

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$x,y$  series and Padé method  
Cartesian coordinate system

**Greek symbols**

$\alpha$  thermal diffusivity  
 $B_1, \beta_2$  undetermined initial constants  
 $\eta$  similarity variable  
 $\theta$  similarity function for temperature  
 $v$  velocity component in y-direction  
 $\psi$  stream function

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**1. Introduction**

Most problems in fluid mechanic and heat transfer are inherently of nonlinearity type. One of these problems is the phenomenon of laminar thermal boundary layer of a heated plate embedded in a fluid flow. This phenomena has a number of thermal engineering applications such as heat exchangers, solar collectors, nuclear reactors, electronic equipments, cooling of metallic plate in a cooling bath, etc.

Velocity distribution in the hydrodynamic boundary layer is given by the well-known Blasius [1] similarity solution. The Blasius problem has attracted considerable interest of many researchers since its introduction in 1908. After Blasius, Sakiadis [2] studied the two dimensional boundary layer flows over a continuously moving flat plate in a quiescent fluid and found exactly the same equation as Blasius with different boundary conditions. This problem was then extended to include blowing or suction at the moving surface by Erickson et al. [3]. Crane [4] investigated flow over a linearly stretching sheet immersed in a quiescent fluid and obtained an exact solution for the Navier-Stokes equation. Then, Gupta and Gupta [5] studied the effect of wall mass transfer over a stretching sheet with suction or blowing. After these pioneering works, the boundary-layer flow passing a flat plate received considerable attention and was studied in different fields [6-8].

The basic studies on similarity solution for the thermal boundary layer over flat plate were well established and widely quoted in heat transfer textbooks such as Incropera et al. [9] and Bejan [10]. Bejan [10] suggested a similarity

temperature variable which reduced energy equation to an ordinary differential equation. Furthermore, various studies have developed a similarity solution for different boundary conditions such as constant surface temperature [11], constant surface heat flux [10], variable heat flux or variable surface temperature [12]. However, when the plate is prescribed to a convective fluid from below, the consideration of constant or variable temperature/heat flux is not a realistic boundary condition in many engineering applications. In this case, the convective boundary condition is more realistic for simulating thermal boundary condition.

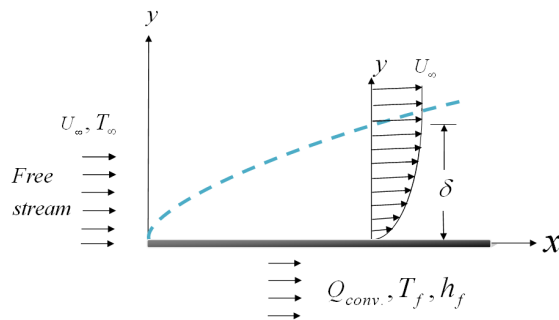
Recently, Aziz [13] studied the classical problem of hydrodynamic and thermal boundary layers over a flat plate with a convective surface boundary condition using similarity solution. In his work, the results were presented for three representative Prandtl numbers of 0.1, 0.72 and 10. Aziz [13] demonstrated that a similarity solution was possible if the convective heat transfer of the fluid heating the plate on its lower surface was proportional to  $x^{-1/2}$ . Ishak [14] developed the work of Aziz [13] by considering effects of suction or injection over the flat surface. Furthermore, in a very recent study as a comment on the work of Aziz [13], Magyari [15] adopted an analytical approach to understand inner features of the heat transfer problem under the convective boundary condition.

In the present study, an integration of Power series-Padé approximation technique (PS- Padé) was utilized to solve the coupled set of governing equations arising from the similarity solution for the laminar thermal boundary layer over a flat plate prescribed convective surface boundary condition. The obtained results were compared with numerical results obtained in the present work as well as the previous works. The paper is organized as follows: governing equation of fluid flow and heat transfer of flat plate embedded in fluid flow is presented in Sections 2. A brief review of power series solution and Padé approximants is given in Sections 3. Mathematical formulation for solving the mentioned equations in Section 2 is discussed in Section 4. The results are discussed in Section 5.

Finally, conclusions and directions for future research are presented in Section 6.

## 2. Governing equations

Consider the steady state two-dimensional laminar boundary layer flow over a static permeable flat plate immersed in a viscous fluid in which temperature of plate is  $T_w$ . The schematic of physical model and coordinate system is depicted in Fig. 1.



**Fig. 1.** Schematic of physical model and coordinate system.

It is assumed that the free stream with constant velocity  $U_\infty$  and constant temperature  $T_\infty$  flows over the thin solid surface. By applying boundary layer assumptions, the basic steady conservation of total mass, momentum and thermal energy in the Cartesian coordinate system of  $x$  and  $y$  can be written as follows [10,14]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left( \frac{\partial^2 u}{\partial y^2} \right) \tag{2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial y^2} \right) \tag{3}$$

subject to the following boundary conditions

$$\begin{aligned} u = 0, \quad v = 0 \quad \text{at} \quad y = 0 \\ u \rightarrow U_\infty \quad \text{at} \quad y \rightarrow \infty \end{aligned} \tag{4}$$

where  $v$  and  $u$  are velocity components in  $y$  and  $x$  directions, respectively.  $T$  is fluid temperature in the boundary layer,  $\alpha$  is thermal diffusivity and  $\nu$  is kinematic viscosity. In Fig. 1, it is assumed that the plate is heated from below by a fluid with heat transfer coefficient  $h_f$  and temperature  $T_f$ . Under this assumption, boundary conditions for the temperature on the plate surface can be written as [13, 14]:

$$-k \frac{\partial T}{\partial y} = h_f (T_f - T_w) \quad \text{at} \quad y = 0 \tag{5}$$

where  $k$  is thermal conductivity of fluid over top surface and  $T_w$  is uniform temperature of the plate where  $T_f > T_w > T_\infty$ .

In order to solve Eqs. (1–5), the following similarity transformations could be introduced [13, 14, and 16]:

$$\begin{aligned} \eta = (U_\infty / \nu x)^{\frac{1}{2}} y, \quad \psi = (U_\infty \nu x)^{\frac{1}{2}} f(\eta), \\ \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \end{aligned} \tag{6}$$

where  $\eta$  is similarity variable,  $f$  is dimensionless stream function,  $\theta$  is dimensionless temperature and  $\psi$  is stream function defined as  $u = \partial\psi/\partial y$  and  $v = -\partial\psi/\partial x$  which identically satisfies continuity equation (i.e. Eq. (1)). Applying similarity transformations to Eqs. (1 - 4) yields,

$$f''' + \frac{1}{2} f f'' = 0 \tag{7}$$

$$\frac{1}{\text{Pr}} \theta'' + \frac{1}{2} f \theta' = 0 \tag{8}$$

subject to the transformed boundary conditions as follows:

$$\begin{aligned}
 f(0) &= 0, & f'(0) &= 0, \\
 \theta'(0) &= -a[1 - \theta(0)] \\
 f'(\eta) &\rightarrow 1, \theta(\eta) \rightarrow 0 \text{ at } \eta \rightarrow \infty
 \end{aligned} \tag{9}$$

where parameter  $a$  is

$$a = \frac{h_f}{k} \sqrt{vx/U_\infty} \tag{10}$$

In order to transform energy equation into a similarity solution, the quantity  $a$  must be a constant and independent from variable  $x$ . Hence, according to Eq. (10), this condition can be met if the heat transfer coefficient  $h_f$  is proportional to  $x^{-1/2}$ . Therefore, it is assumed that

$$h_f = cx^{-1/2} \tag{11}$$

where  $c$  is a constant. By substituting Eq. (11) in Eq. (10), the following equation can be obtained

$$a = \frac{c}{k} \sqrt{v/U_\infty} \tag{12}$$

Consider that the case of  $a \rightarrow \infty$  reduces the present study to the classical thermal boundary layer flow over a flat plate as considered by Pohlhausen [17]. The case of  $a=0$  represents a plate with adiabatic boundary condition in which the thermal boundary layer no longer exists.

The value of local Nusselt number as an important parameter for heat transfer is given by,

$$Nu_x = \theta'(0) Re_x^{\frac{1}{2}} \tag{13}$$

Therefore, the values of  $Nur = Nu_x Re_x^{-\frac{1}{2}}$  depend on the evaluated value of  $\theta'(0)$  from Eq. (8).

### 3. Solution method

#### 3.1. Power series solution

A system of first order differential equations can be considered as:

$$\begin{cases}
 y'_1 = f_1(x, y_1, \dots, y_n) \\
 y'_2 = f_2(x, y_1, \dots, y_n) \\
 \vdots \\
 y'_n = f_n(x, y_1, \dots, y_n)
 \end{cases} \tag{14}$$

with initial condition of  $y_i(x_0) = y_{0,i}$ ,  $i=1, 2, \dots, n$  where  $f$  and  $y$  are vector functions. The solutions of Eq. (14) can be assumed such that:

$$y = y_0 + ex \tag{15}$$

where  $e$  is a vector function. By substituting Eq. (15) in Eq. (14) and neglecting the higher order terms, the linear equation of  $e$  can be obtained as:

$$Ae = B \tag{16}$$

where  $A$  and  $B$  are constant matrices. By solving Eq. (16), coefficients of  $e$  in Eq. (15) can be determined. By repeating the above procedure for higher terms, the arbitrary order power series of the solutions can be obtained for Eq. (14). For more details, the readers can refer to [18-23]. A power series can be defined in the form of:

$$\begin{aligned}
 f(x) &= f_0 + f_1 x + f_2 x^2 + \dots + \\
 &(f_n + p_1 e_1 + \dots + p_m e_m) x^n
 \end{aligned} \tag{17}$$

where  $p_1, p_2, \dots, p_m$  are constants and  $e_1, e_2, \dots, e_m$  are bases of vector  $e$ ;  $m$  is size of vector  $e$  and  $y$  is a vector in Eq. (14) with  $m$  elements. Every element can be represented by the power series in Eq. (17):

$$y_i = y_{i,0} + y_{i,1} x + y_{i,2} x^2 + \dots + e_i x^n \tag{18}$$

where  $y_i$  is  $i$ th element of  $y$ . By substituting Eq. (18) in Eq. (1), the following can be found:

$$f_i = (f_{i,n} + p_{i,1}e_1 + \dots + p_{i,m}e_m)x^{n-j} + Q(x^{n-j+1}) \tag{19}$$

where  $f_i$  is  $i$ th element of  $f(y, y', x)$  in Eq. (7) and if  $f(y, y', x)$  have  $y'$  then  $j$  is 0; otherwise, 1.

From Eqs. (19) and (16) and according to Eq. (16), the following linear equations can be determined:

$$A_{i,j} = P_{i,j}, \quad B_i = -f_{i,n} \tag{20}$$

Solving this linear equation determines  $e_i$  ( $i=1, \dots, m$ ). By substituting  $e_i$  in Eq. (18),  $y_i$  ( $i=1, \dots, m$ ) can be presented which are polynomials of degree  $n$ . Repeating this procedure from Eqs. (18-20) results in getting the arbitrary order power series solution of differential-algebraic equations of Eq. (14). For more details, the readers may refer to [24-26].

### 3.2 The Padé approximation

As is known, every power series represents a function  $f(x)$  so that:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \tag{21}$$

This expansion is the fundamental starting point of any analysis using Padé approximants. The objective of the Padé approximants is to seek a rational function for the series since Padé approximants converge on the entire real axis if the series solution is free from singularities on the real axis; moreover, it provides more information on the mathematical behavior of the solution. A Padé approximant is a rational fraction. The notation for such a Padé approximant is:

$$a_0 + a_1x + a_2x^2 + \dots = \frac{p_0 + p_1x + p_2x^2 + \dots + p_Mx^M}{1 + q_1x + q_2x^2 + \dots + q_Lx^L} \tag{22}$$

By cross-multiplying Eq. (22) and comparing coefficients of both sides, it can be found that:

$$a_l + \sum_{k=1}^M a_{l-k}q_k = p_l, \quad l = 0, \dots, M \tag{23}$$

$$a_l + \sum_{k=1}^L a_{l-k}q_k = 0, \quad l = M + 1, \dots, M + L \tag{24}$$

By solving linear Eq. (24),  $q_k$  ( $k=1, \dots, L$ ) is determined. Now, by substituting  $q_k$  in Eq.(23),  $p_l$  ( $l=0, \dots, M$ ) is determined. The coefficients in Eq. (24) are Toeplitz matrix. Therefore, the above linear equations can be easily solved using Gaussian elimination if  $M \leq L \leq M + 2$ , where  $M$  and  $L$  are degrees of numerator and denominator in Padé series, respectively. After that, this Padé series gives an A-stable formula for an ordinary differential equation [16, 17]. For more information, the readers can refer to [21-23, 27-30].

### 4. Mathematical formulation

Equations (7) and (8) can be written as the following set of first order nonlinear differential equations:

$$\begin{cases} y_1'(\eta) - y_2(\eta) = 0, \\ y_2'(\eta) - y_3(\eta) = 0, \\ y_3'(\eta) + \frac{y_1(\eta)y_3(\eta)}{2} = 0, \\ y_4'(\eta) - y_5(\eta) = 0, \\ y_5'(\eta) + \Pr \frac{y_1(\eta)y_5(\eta)}{2} = 0, \end{cases} \quad 0 \leq \eta \tag{25-a}$$

subject to:

$$y(0) = \begin{pmatrix} 0 \\ 0 \\ \beta_1 \\ \beta_2 \\ -a(1-\beta_2) \end{pmatrix} \tag{25-b}$$

with the following constraints:

$$y_2(\infty) = 1 \qquad y_4(\infty) = 0 \tag{26}$$

where  $\beta_1$  and  $\beta_2$  are constants which will be computed from the boundary condition Eq. (26). By solving Eq. (25-a) with the boundary conditions (25-b) and then applying the related boundary conditions (i.e.  $y_2(\infty) = 0$  and

$y_4(\infty) = 0$ ), the values of  $\beta_1$  and  $\beta_2$  can be determined. Based on the power series method introduced in the previous section, the solution procedure can be started as follows:

$$\begin{aligned} y_1(\eta) &= 0 + e_1\eta, & y_2(\eta) &= 0 + e_2\eta, \\ y_3(\eta) &= \beta_1 + e_3\eta, & y_4(\eta) &= \beta_2 + e_4\eta, \\ y_5(\eta) &= a(1 - \beta_2) + e_5\eta, \end{aligned} \quad (27)$$

Substituting Eq. (27) in Eq. (25-a) and neglecting higher order terms yield:

$$\begin{aligned} e_1 - 0 &= 0, & e_2 - \beta_1 &= 0, & e_3 - 0 &= 0, \\ e_4 - a(\beta_2 - 1) &= 0, & e_5 - 0 &= 0, \end{aligned} \quad (28)$$

Solving Eq. (28) for  $e_1$  to  $e_5$ :

$$\begin{aligned} e_1 &= 0, & e_2 &= \beta_1, & e_3 &= 0, \\ e_4 &= a(\beta_2 - 1), & e_5 &= 0, \end{aligned} \quad (29)$$

Now, substituting  $e_1$  to  $e_5$  in Eq. (27) and considering a higher order term yield:

$$\begin{aligned} y_1(\eta) &= e_1\eta^2, & y_2(\eta) &= \beta_1\eta + e_2\eta^2, \\ y_3(\eta) &= \beta_1 + e_3\eta^2, \\ y_4(\eta) &= \beta_2 + a(\beta_2 - 1)\eta + e_4\eta^2, \\ y_5(\eta) &= a(1 - \beta_2) + e_5\eta^2, \end{aligned} \quad (30)$$

Again, substituting Eq. (30) in Eq. (25-a) and neglecting higher order terms yield:

$$\begin{aligned} 2e_1 - \beta_1 &= 0, & e_2 &= 0, & e_3 &= 0, \\ e_4 &= 0, & e_5 &= 0, \end{aligned} \quad (31)$$

Solving it for  $e_1$  and  $e_5$ , substituting the obtained values of  $e$  in Eq. (27) and considering a higher term could lead to:

$$\begin{aligned} y_1(\eta) &= \frac{\beta_1}{2}\eta^2 + e_1\eta^3, \\ y_2(\eta) &= \beta_1\eta + e_2\eta^3, \\ y_3(\eta) &= \beta_1 + e_3\eta^3, \\ y_4(\eta) &= \beta_2 + a(\beta_2 - 1)\eta + e_4\eta^3, \\ y_5(\eta) &= a(1 - \beta_2) + e_5\eta^3, \end{aligned} \quad (32)$$

By substituting Eq. (32) in Eq. (25-a) and repeating this procedure, the following power series can be obtained for  $y_1$ ,  $y_4$  and  $y_5$  after eight iterations:

$$\begin{aligned} y_1(\eta) &= \frac{1}{2}\beta_1\eta^2 - \frac{1}{2 \times 5!}\beta_1^2\eta^5 + \\ &\frac{11}{4 \times 8!}\beta_1^3\eta^8 + O(\eta^8), \end{aligned} \quad (33-a)$$

$$\begin{aligned} y_4(\eta) &= \beta_2 + a(\beta_2 - 1)\eta \\ &+ \text{Pr} \beta_1^2 a(\beta_2 - 1) \left( \frac{1}{240} + \frac{\text{Pr}}{24} \right) \eta^7 \\ &- \text{Pr} \frac{\beta_1 a(\beta_2 - 1)}{2 \times 4!} \eta^4 + O(\eta^8) \end{aligned} \quad (33-b)$$

$$\begin{aligned} y_5(\eta) &= \frac{1}{12} \text{Pr} a \beta_1 (1 - \beta_2) \eta^3 \\ &- a(1 - \beta_2) + 7 \text{Pr} a \beta_1^2 (1 - \beta_2) \\ &\cdot \left( -\frac{1}{20160} - \frac{1}{2016} \text{Pr} \right) \eta^6 + O(\eta^8) \end{aligned} \quad (33-c)$$

Although the obtained power series (i.e. Eq. (33)) is sufficiently accurate, the series solutions usually have a finite range of convergence. Hence, the obtained power series in the symbolic form (before computation of the unknown values of  $\beta_1$  and  $\beta_2$ ) can be converted into Padé approximation in order to increase accuracy of the solution. For instance, by following the introduced procedure in the previous section, the Padé series of Eq. (33) with the size of  $\{3, 3\}$  can be obtained as follows:

$$y_1(\eta) = \frac{60\beta_1\eta^2}{\beta_1\eta^3 + 120} \quad (34-a)$$

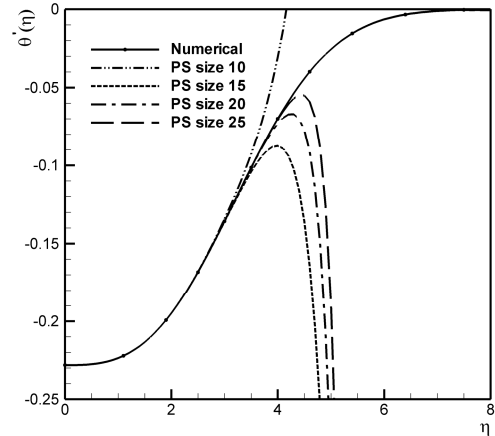
$$y_4(\eta) = \left[ \frac{\beta_2 + a \times (\beta_2 - 1)\eta + \frac{1}{48} Pr \times \beta_1 \beta_2 \eta^3}{1 + \frac{1}{48} Pr \times \beta_1 \eta^3} \right] \quad (34-b)$$

$$y_5(\eta) = \left[ \frac{a(\beta_2 - 1) - \left( \frac{1}{24} a Pr \beta_1 (\beta_2 - 1) + \frac{1}{240} a \beta_1 (1 - \beta_2) \right) \eta^3}{1 + \frac{1}{24} \beta_1 \left( \frac{1}{10} + Pr \right) \eta^3} \right] \quad (34-c)$$

**5. Results and discussion**

In order to demonstrate accuracy of the presented method, a fair comparison was made between the presented solution and numerical results. Numerical results were obtained using Maple mathematical software. Maple uses a combination of trapezoid as a basic scheme and Richardson extrapolation as an enhancement scheme, as introduced in [31, 32]. The Maple commands which were used to obtain the numerical results are shown in Appendix A. The presented method (PS - Padé) reduced computational difficulties of other methods (the same as the HAM [33,34]). A comparison between the obtained values of  $\theta'(\eta)$  by power series method (for various series sizes) and the numerical method is shown in Fig. 2 when  $Pr=0.72$  and  $a=1$ . As seen in this figure, the obtained results using power series method were in good agreement with numerical results; however, this series diverged around infinity. Figure 2 confirms that the series solution, even with 25 terms ( $x^{25}$ ), diverged before the function reached its asymptotic value of infinity. Here, the Padé approximants could be utilized to enhance convergence range of the solution. Combinations of power series solutions with the Padé approximants increased the convergence of given series. The {3, 3}, {5, 5}, {7, 7} {9, 9} and {11,11} Padé

approximants for  $Pr=0.72$  and  $a=1$  were compared with numerical results in Fig. 3. This figure shows that the Padé approximants with size of {9, 9} was sufficiently accurate. Hence, Padé approximants with size of {9, 9} were selected in the following calculations for convenience.



**Fig. 2.** Comparing power series results ( $y_5$ ) with numerical results for  $Pr =0.72$  and  $a=1$  for different series sizes.

For  $Pr =0.72$  and  $a=1$ , the following power series solution was obtained:

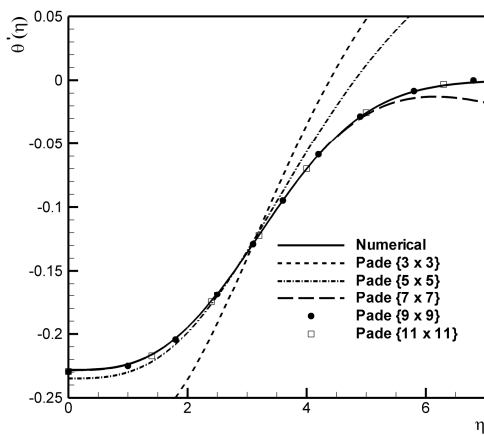
$$f(\eta) = 0.16699860100\eta^2 - 0.4648088792 \times 10^{-3}\eta^5 + 0.25412106 \times 10^{-5}\eta^8 - 0.146136 \times 10^{-7}\eta^{11} + 0.831 \times 10^{-10}\eta^{14} - 0.5 \times 10^{-12}\eta^{17} \quad (35-a)$$

$$f'(\eta) = 0.33339972021000\eta - 0.23240443960 \times 10^{-2}\eta^4 + 0.203296847 \times 10^{-4}\eta^7 - 0.1607495 \times 10^{-6}\eta^{10} + 0.11638 \times 10^{-8}\eta^{13} - 0.79 \times 10^{-11}\eta^{16} \quad (35-b)$$

$$\begin{aligned} \theta(\eta) = & 0.7705316513000 \\ & -0.2294683487000\eta \\ & +0.11496267970 \times 10^{-2} \eta^4 \\ & +0.74965988 \times 10^{-5} \eta^7 \\ & +0.459360 \times 10^{-7} \eta^{10} \\ & -0.2680 \times 10^{-9} \eta^{13} \\ & +0.15 \times 10^{-11} \eta^{16} \end{aligned} \quad (35-c)$$

$$\begin{aligned} \theta'(\eta) = & 0.2294683487000 \\ & +0.45985071860 \times 10^{-2} \eta^3 \\ & -0.524761916 \times 10^{-4} \eta^6 \\ & +0.4593602 \times 10^{-6} \eta^9 \\ & -0.34843 \times 10^{-8} \eta^{12} \\ & +0.244 \times 10^{-10} \eta^{15} \\ & -0.2 \times 10^{-12} \eta^{18} \end{aligned} \quad (35-d)$$

The obtained equations (i.e. Eq. (35)) had sufficient accuracy but series solutions usually have a finite range of convergence and hence are not always practical for large values of  $\eta$ , say  $\eta \rightarrow \infty$ .



**Fig. 3.** Comparing the Padé results ( $y_5$ ) with numerical results for  $Pr = 0.72$  and  $a=1$  and series size of 20 for different Padé sizes.

As mentioned before, the combination of any series solutions with the Padé approximants provided a powerful tool for handling initial or boundary value problems on infinite or semi-infinite domains [30]. If the order of Padé approximation increases, the accuracy of

solution increases [29]. As seen in Fig. 3, the order of Padé approximation  $\{9, 9\}$  had sufficient accuracy. The Padé approximant for Eq. (35) with size of  $\{9, 9\}$  was as follows:

$$f'(\eta) = \frac{\begin{bmatrix} (114789608.70\eta^7 \\ +32563000180.00\eta^4 \\ +5622567066000.00\eta) \end{bmatrix}}{\begin{bmatrix} (691489.31\eta^9 \\ +812490706.83\eta^6 \\ +214631643070.73\eta^3 \\ +16834174156800.00) \end{bmatrix}} \quad (36-a)$$

$$f(\eta) = \frac{\begin{bmatrix} (438379810.70\eta^8 \\ +199189570700.00\eta^5 \\ +21694626350000.00\eta^2) \end{bmatrix}}{\begin{bmatrix} (1561074.74\eta^9 \\ +4974437980.00\eta^6 \\ +1554339018000.00\eta^3 \\ +129909030400000) \end{bmatrix}} \quad (36-b)$$

$$\theta(\eta) = \frac{\begin{bmatrix} (3515584875\eta^9 \\ -808702564500\eta^7 \\ +6861883840000\eta^6 \\ -380390140400000\eta^4 \\ +2768395096000000\eta^3 \\ -8863395065000000\eta \\ +297623897800000000) \end{bmatrix}}{\begin{bmatrix} (4562544406\eta^9 \\ +8905388670000\eta^6 \\ +3592837610000000\eta^3 \\ +386257848400000000) \end{bmatrix}} \quad (36-c)$$

$$\theta'(\eta) = \frac{\begin{bmatrix} (670754861.00\eta^9 \\ -505162836500.00\eta^6 \\ +171970682600000.00\eta^3 \\ -25142829930000000.00) \end{bmatrix}}{\begin{bmatrix} (8479158670.00\eta^9 \\ +6128594105000.00\eta^6 \\ +1446331968000000.00\eta^3 \\ +109569925800000000.00) \end{bmatrix}} \quad (36-d)$$



Figure 4 shows variation of velocity profiles ( $f'(\eta)$ ) with  $\eta$ . In this figure, numerical results are compared with PS – Padé results. In this case, the initial slop (i.e.  $\beta_1$ ) was obtained as 0.333997, where a highly accurate numerical solution of Blasius equation was  $f''(0) = 0.332057$ , as provided by Howarth [35, 36]. Table 1 compares stream line solution (i.e.  $f(\eta)$ ) obtained by the presented work with those reported by Ahmad et al. [37] and Lin [38]. As can be seen, there was good agreement between results of PS – Padé and those of previous works. The values of reduced Nusselt number ( $Nu_x/\sqrt{Re_x}$ ) were provided by analytical solution using Padé approximation. According to Eq. (13), the values of reduced Nusselt number were equal to  $\theta'(0)$ .

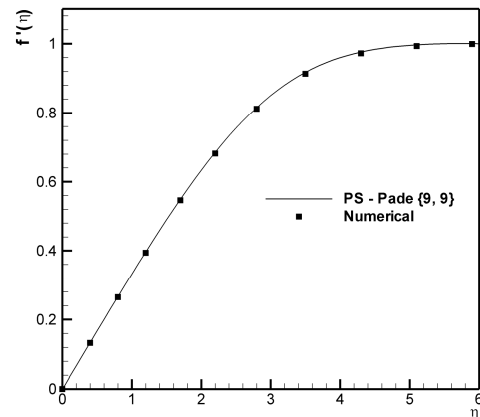
**Table 1.** Comparing  $f(\eta)$  obtained using PS - Padé method in the present study and previous works.

$\eta$	Present work PS-Padé {9, 9}	Ahmad [37]	Lin [38]
0	0	0	0
0.4	0.026715	0.0266	-
0.8	0.106727	0.1061	-
1.2	0.239332	0.2379	-
1.6	0.422749	0.4203	-
2	0.653742	0.65	0.650101
2.4	0.927492	0.9223	-
2.8	1.237797	1.2311	-
3.2	1.577606	1.5693	-
3.6	1.93975	1.9297	-
4	2.317668	2.3058	2.305726
4.4	2.705933	2.6922	-
4.8	3.100462	3.0848	-
5.2	3.498416	3.4813	-
5.6	3.897907	3.8799	-
6	4.297634	4.2794	4.27973
6.4	4.696541	4.6793	-
6.8	5.093567	5.0792	-

Tables 2 and 3 show values of  $\theta(0)$  and  $\theta'(0)$ , respectively, for three selected values of Prandtl number as 0.1, 0.72 and 10 and for different values of parameter  $a$ . In these tables, a comparison was made between PS-Padé solution and numerical solution reported by Aziz [13] and Ishak [20]. As mentioned, case of  $a \rightarrow \infty$  simulated an isothermal flat plate which

was analyzed in the work of Soundalgekar [39] and Wilks [40].

A comparison between the results of present study and previous ones is given in Table 4. The results of this table show excellent agreement between the analytical and numerical results as well as the results reported by previous researchers.



**Fig. 4.** Variation of  $f'(\eta)$  with  $\eta$  obtained by Power series - Padé and numerical solution.

**Table 2.** Values of  $-\theta'(0)$  for various values of  $a$  when  $Pr = 10$ .

$a$	$-\theta'(0)$		
	Aziz [20]	Present results numerical	Present results PS-Pade {9, 9}
0.05	0.04688	0.04678	0.04681
0.1	0.08792	0.08792	0.08798
0.2	0.15690	0.15690	0.15708
0.4	0.25827	0.25817	0.25865
0.6	0.32894	0.32894	0.32972
0.8	0.38129	0.38119	0.38224
1	0.42134	0.42134	0.42262
5	0.63568	0.63558	0.63850
10	0.67872	0.67872	0.68205
20	0.70266	0.70256	0.70613

Variations of  $\theta(\eta)$  and  $\theta'(\eta)$  against variation of  $\eta$  for various values of  $a$  with  $Pr = 0.72$  are shown in Figs. 5 and 6, respectively. Moreover, in these figures, numerical results are compared with PS – Padé results. As can be observed, increase of parameter  $a$  increases  $\theta'(0)$  (i.e. Nusselt number) and  $\theta(0)$ . These figures indicate that analytical

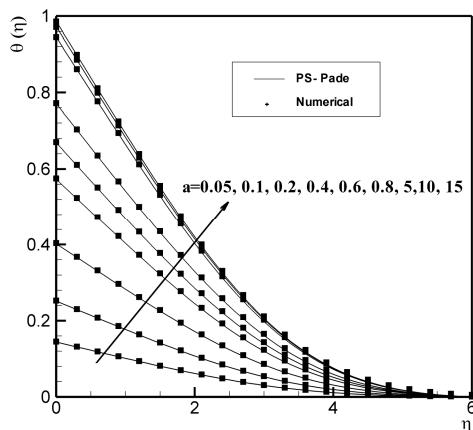
results coincide with numerical results for all values of parameter  $a$ .

**Table 3.** Values of  $\theta(0)$  for various values of  $a$  when  $Pr = 10$ .

$a$	$\theta(0)$		
	Aziz [20]	Present result numerical	Present result analytical
0.05	0.0643	0.06426	0.06379
0.1	0.1208	0.12075	0.12019
0.2	0.2155	0.21548	0.21459
0.4	0.3546	0.35456	0.35336
0.6	0.4518	0.45175	0.45045
0.8	0.5235	0.52351	0.52220
1	0.5787	0.57865	0.57737
5	0.8729	0.87288	0.87230
10	0.9321	0.93212	0.93179
20	0.9649	0.96487	0.96469

**Table 4.** Values of  $-\theta'(0)$  for  $a \rightarrow \infty$ .

	Pr=0.1	Pr=0.72	Pr=10
Soundalgekar [39]	0.1472	-	-
Wilks [40]	-	0.2965	0.7281
Present results numerical	0.14718	0.29563	0.72813
Present results analytical Padé {9, 9}	0.14708	0.2955	0.72811



**Fig. 5.** Variation of  $\theta(\eta)$  with  $\eta$  obtained by Power series – Padé {9, 9} and numerical solution for various values of  $a$ .

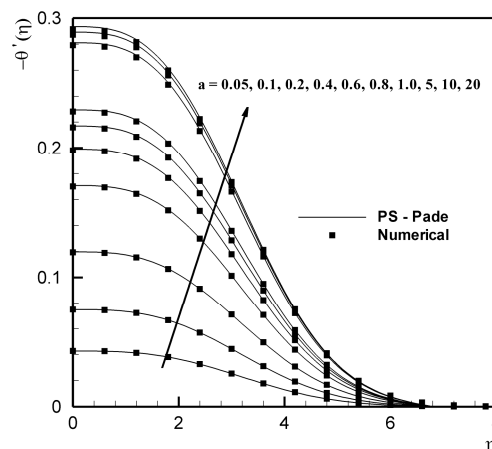
### 6. Conclusions

Nonlinear differential equation arising from similarity solution of flow and heat transfer of a viscous fluid over a flat plate prescribed to

convective boundary condition was examined using combination of a symbolic power series and Padé approximation method. The present PS-Padé not only was very simple and effective but also could be used to develop a simple analytic solution for flow and heat transfer over the flat plate. The obtained analytical solution was in non-dimensional form; hence, its results could be easily utilized for different fluids and thermal boundary conditions in practical applications. The results of present study can be summarized as follows:

- 1- The method was successfully applied directly to solve the two point boundary value governing differential equation without requiring discretization, linearization or perturbation.
- 2- The value of  $f'(0)$  was determined as 0.333997 using analytical solution which indicated sufficient accuracy in comparison with accurate numerical value of 0.332057.
- 3-  $\theta(0)$  and  $\theta'(0)$ , as important parameters of heat transfer in the boundary layer, were calculated by the obtained analytical solution for different values of parameter  $a$ .
- 4- The presented solution represented remarkable accuracy when the results were compared with the numerical ones.

The future works can be focused on comparing effect of changing power series technique on the accuracy of solution.



**Fig. 6.** Variation of  $\theta'(\eta)$  with  $\eta$  obtained by Power series – Padé {9, 9} and numerical solution for different values of  $a$ .

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**Appendix A**

Maple code:

```
> Pr := .72; a:=0.05;
> sys:= 2*(diff(y(x), x, x, x)) = -y(x)*(diff(y(x),
x, x)), (diff(z(x), x,
x))/(Pr)+(1/2)*y(x)*(diff(z(x), x));
> fcn := {y(x), z(x)};
> p:=dsolve({sys,
y(0)=0,z(10)=0,(D(y))(0)=0,(D(y))(10)=1,(D(z))
(0)=-a*(1-z(0))}, fcn, type = numeric);
> odeplot(p, [x, z(x)], 0 .. 10, numpoints =
25);
> subs(p(0), -(diff(z(x), x)));
> subs(p(0), z(x));
> for b from 0 by .1 to 10 do evalf(subs(p(b), -(diff(z(x), x)))) end do;
> for b from 0 by .1 to 10 do evalf(subs(p(b), z(x))) end do;
```

In lines one and two, the fixed values of Pr and parameter of a as well as system of differential equations are defined. In line 3, the command of dsolve from the package of Maple is called to solve system of differential equations subject to the boundary conditions. Finally, the remaining lines are used to plot and extract data from the results of dsolve command.

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